PARTIAL DIFFERENTIAL EQUATIONS M-304

UNIT - I

Introduction, Methods of Solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, Orthogonal trajectories of a System of curves on a Surface, pfaffin differential forms and Equations, Solutions of pfaffin Differential equations in three variables, Cauchy's problem for first order partial differential Equations.

Sec - 3 to 6 of chapter I & Sec - 1, 2, 3 of chapter II

UNIT - II

Linear Equations of the first order, Integral Surfaces, Orthogonal Surfaces, Non-linear partial differential equations of the first order, Cauchy's method of characteristics, Compatible Systems of first order Equations, charpit's method, Special types of first order Equations, Jacobi's method.

Sec - 4 to 13 of chapter II

UNIT - III

Partial Differential Equations of the second order, Their origin, Linear partial Differential Equations with constant and variable coefficients, Solutions of linear hyperbolic Equations, Method of Separation of variables, Monger's method

Sec - 1 to 5 & sec - 8, 9, 11 of chapter III

UNIT - IV

Laplace Equation, Elementary solutions, Families of Equipotential Surfaces, Boundary Value Problems, Method of Separation of Variables of solving Laplace Equation, Problems with axial Symmetry, Kelvin's inversion Theorem, the wave Equation, Elementary solution in One-dimensional form, Riemann-Volterra Solution of one-dimensional wave equation.

Sec - 1 to 7 of chapter IV & sec - 1 to 3 of chapter V

Textbook:- Elements of partial Differential Equations

-I.N. Sneddon

M.C. Grawhill, International Edition, Mathematics, Serie

Reference:- Fritz John, partial Differential Equations, Narosa

Publishing house, NewDelhi - 1979

UNIT-I

Introduction:-

Differential Equation:- An equation involving derivatives (or) differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Ex:- 1, $dy = (x + \sin x)\,dx$

2, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^t$

3, $y = \sqrt{x}\,\dfrac{dy}{dx} + \dfrac{k}{dy/dx}$

This eq^n can also be written as $y = \sqrt{x}\,y_1 + \dfrac{k}{y_1}$ where $y_1 = \dfrac{dy}{dx}$

Ordinary differential equation:- A differential equation involving derivatives with respect to a single independent variable is called a ordinary differential equation.

Ex:- 1, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^t$

2, $k\left(\dfrac{d^2y}{dx^2}\right) = \left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}$

Partial Differential Equation:- A differential equation involving partial derivatives with respect to one or more independent variables is called a "partial Differential Equation"

Ex:- 1, $\dfrac{\partial^2 v}{\partial t^2} = k\left(\dfrac{\partial^2 v}{\partial x^3}\right)^2$

2, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

Order of a differential Equation:- The order of a differential equation is the order of the highest derivative occurring in the equation.

Ex:- 1, $(x^2+1)\dfrac{dy}{dx} + 2xy = 4x^2$

order of above eq^n is 1.

2, $\left(\dfrac{dy}{dx}\right)^2 + 8xy = 4x^2$

order of above Differential eq^n is 2.

order of above

Degree of a differential equation:- The degree of a differential equation is the degree of the highest derivative occuring in the differential equation after the equation has been expressed in the form free from radicals and fractions as far as the derivatives are concerned.

Ex:- 1, $\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 12y = 0$

degree of the above D.E is '3'

2, $a\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

squaring on both sides

$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

degree of the above D.E is '2'.

Linear and Non-linear differential equations:-

A differential equation is called linear if every dependent variable and every derivative occuring in the equation is of first degree only and no product of dependent variables or derivatives occur.

→ A differential equation is not linear is called non-linear differential equation.

Ex:- 1, $\frac{dy}{dx} = x + \sin x$ is a linear differential equation.

2, $\sqrt{y} = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx}$

3, $\sqrt{y} = \dfrac{\sqrt{x}\left(\frac{dy}{dx}\right)^2 + k}{\left(\frac{dy}{dx}\right)}$

$\Rightarrow \sqrt{y}\left(\frac{dy}{dx}\right) = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$ is a non-linear differential eq^n.

Sec-3 Methods of Solution of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

→ We know that the integral curves of the set of differential equations is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ —(1) form a two parameter family of curves in three-dimensional space.

→ from eqⁿ (1) we can derive two relations of the form

$u_1(x,y,z)=c_1$ and $u_2(x,y,z)=c_2$ —(2)

→ Involving two arbitrary constants c_1 and c_2

→ By varying these constants we obtain a two parameter family of curves satisfying the differential eqⁿ (1)

→ There are three methods to solve such type of differential eqⁿs

Method:- I

i, solve $xzp+yzq=xy$

Given eqⁿ is $xzp+yzq=xy$ —(1)

compare ① with $Pp+Qq=R$

where $P=xz$; $Q=yz$; $R=xy$

we know that integral form of eqⁿ(1) is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Substitude P, Q, R in above eqⁿ

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy}$$

Now taking first two terms

$$\frac{dx}{x\cancel{z}}=\frac{dy}{y\cancel{z}}$$

$$\Rightarrow \frac{dx}{x}=\frac{dy}{y}$$

Integrating on both sides

$\Rightarrow \log x=\log y+\log c_1$

$\Rightarrow x=yc_1$ —(3)

Now taking second and third functions

$$\frac{dy}{yz}=\frac{dz}{xy}$$

$\Rightarrow x\,dy=z\,dz$

Substitude 'x' value in above eqⁿ

$\Rightarrow y c_1 \, dy = z \, dz$

Integrating on both sides

$$\Rightarrow c_1 \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$\Rightarrow c_1 y^2 = z^2 + c_2 \quad —(4)$$

$\therefore$ (3) and (4) are required soln's of given eqn

ii. Solve $xp - yq = xy$

Given eqn is $xp - yq = xy \sim (1)$

compare (1) with $Pp + Qq = R$

where $P = x$; $Q = -y$; $R = xy$

W.K.T integral eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Substitude P, Q, R in above eqn

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{xy}$$

Now take first two terms

$$\frac{dx}{x} = \frac{dy}{-y}$$

$$\log x = -\log y + \log c_1$$

$$x = \frac{c_1}{y} \quad -(2)$$

Now taking second and third terms

$$\frac{dy}{-y} = \frac{dz}{xy}$$

$$-dy = dz\left(\frac{y}{c_1}\right)$$

$$\left(\frac{c_1}{y}\right)(-dy) = dz\left(\frac{y}{xy}\right)$$

Integrating on both sides

$$-c_1 \log y = z + c_2$$

Sub. c_1 value in above eqn

$$(-xy) \log y = z + c_2$$

$$z + c_2 + (xy) \log y = 0 \sim (3)$$

$\therefore$ (2) and (3) are req. soln's of given eqn

iii) Solve $py + qx = xyz^2(x^2-y^2)$

Given eqⁿ is $py+qx = xyz^2(x^2-y^2)$ ---(1)

compare (1) with $Pp+Qq=R$

where $P=y$; $Q=x$; $R = xyz^2(x^2-y^2)$

W.K.T Lagrange's subsidary eqⁿ of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Substitude P, Q, R in above eqⁿ

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$$

Now take first two terms

$$\frac{dx}{y} = \frac{dy}{x}$$

$$x\,dx = y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1}{2}$$

$$x^2 = y^2 + c_1 \quad -(2)$$

Now take second and third terms

$$\frac{dy}{\cancel{x}} = \frac{dz}{\cancel{x}yz^2(x^2-y^2)}$$

$$dy = \frac{dz}{yz^2c_1}$$

$$c_1 y\,dy = \frac{dz}{z^2}$$ Integrating on both sides

$$c_1\frac{y^2}{2} = -\frac{1}{z} + c_2$$

$$c_2 = \frac{c_1 y^2}{2} + \frac{1}{z}$$ substitude $c_1 = x^2 - y^2$ in above eqⁿ

$$c_2 = \frac{(x^2-y^2)y^2}{2} + \frac{1}{z} \quad -(3)$$

∴ (3) and (4) are req. solⁿ's of given eqⁿ.

iv) Solve $p+3q = 5z + \tan(y-3x)$

Given eqⁿ is $p+3q = 5z+\tan(y-3x)$ ---(1)

compare (1) with $Pp+Qq=R$

where $P=1$; $Q=3$; $R = \tan(y-3x)+5z$

W.K.T. Lagrange's subsidary eqⁿ of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Substitude P, Q, R in above eq^n

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

Now taking first and second terms

$$dx = \frac{dy}{3}$$

Integrating on both sides

$$x = \frac{y}{3} + c_1$$

$$3x = y + 3c_1$$

$$3c_1 = 3x - y \Rightarrow y - 3x = -3c_1$$

$$c_1 = \frac{3x-y}{3} \Rightarrow \frac{y-3x}{3} = -c_1 \sim (2)$$

Now taking first and third terms

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)}$$

$$dx = \frac{dz}{5z + \tan(-3c_1)}$$

Integrating on both sides

$$x = \frac{\log(5z + \tan(-3c_1))}{5} + c_2$$

$$5x = \log(5z + \tan(-3c_1) + c_2$$

$$5x = \log(5z + \tan(\cancel{3}(\frac{y-3x}{\cancel{3}})) + c_2$$

$$5x = \log(5z + \tan(y-3x)) + c_2 \quad -(3)$$

$\therefore$ (2) and (3) are req^n sol^n's of given eq^n

Method-II :-

i, Solve $z(x+y)p + z(x-y)q = x^2+y^2$

Given eq^n is $z(x+y)p + z(x-y)q = x^2+y^2 \sim (1)$

Compare (1) with $Pp + Qq = R$

where $P = z(x+y)$; $Q = z(x-y)$; $R = x^2+y^2$

W.K.T. Lagrange's subsidary eq^n of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

choose $x, -y, -z$ ap multipliers of each of above functions

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = 0$$

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{x^2\cancel{z} + \cancel{xyz} - \cancel{xyz} + \cancel{y^2z} - \cancel{x^2z} - \cancel{y^2z}} = 0$$

$$x\,dx - y\,dy - z\,dz = 0$$

Integrating on both sides

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_1}{2}$$

$$x^2 - y^2 - z^2 = c_1 \quad \text{— (2)}$$

choose $y, x, -z$ as multipliers of each of above $\overset{\text{fun}}{\cancel{eq}}$ct's, we get

$$\frac{y\,dx + x\,dy - z\,dz}{zy(x+y) + xz(x-y) - z(x^2+y^2)} = 0$$

$$\frac{y\,dx + x\,dy - z\,dz}{\cancel{xzy} + \cancel{zy^2} + \cancel{x^2z} - \cancel{xzy} - \cancel{zx^2} - \cancel{zy^2}} = 0$$

$$y\,dx + x\,dy - z\,dz = 0$$

Integrating on both sides

$$d(xy) - z\,dz = 0$$

Integrating on both sides

$$xy - \frac{z^2}{2} = \frac{c_2}{2}$$

$$2xy - z^2 = c_2 \quad \sim (3)$$

$\therefore$ 2 and 3 are req. soln's of given eqn.

ii, Solve $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Given eqn is $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$ — (1)

Compare (1) with $Pp + Qq = R$

where $P = x(y^2-z^2)$; $Q = -y(z^2+x^2)$; $R = z(x^2+y^2)$

W.K.T. Lagrange's subsidary eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

choose x, y, z as multipliers of each of above $\overset{\text{fun's}}{\text{eqn}}$, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} - \cancel{y^2z^2} - \cancel{x^2y^2} + \cancel{z^2x^2} + \cancel{z^2y^2}} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating on both sides

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 + z^2 = c_1 \quad \text{--- (2)}$$

Choose $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers of each of above fun's, we

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{\frac{1}{\cancel{x}}\cancel{x}(y^2-z^2) + \frac{1}{\cancel{y}}\cancel{y}(z^2+x^2) - \frac{1}{\cancel{z}}\cancel{z}(x^2+y^2)} = 0$$

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{\cancel{y^2} - \cancel{z^2} + \cancel{z^2} + \cancel{x^2} - \cancel{x^2} - \cancel{y^2}} = 0$$

$$\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$$

Integrating on bothsides

$$\log x - \log y - \log z = \log c_2$$

$$\log x - (\log y + \log z) = \log c_2$$

$$\log x - (\log yz) = \log c_2$$

$$\log\left(\frac{x}{yz}\right) = \log c_2$$

$$\frac{x}{yz} = c_2 \quad \text{--- (3)}$$

∴ (2) and (3) are req. soln's of given equation

Ex: Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Given eqn is $(mz-ny)p + (nx-lz)q = ly-mx$ --- (1)

Compare (1) with $Pp + Qq = R$

where $P = mz-ny$; $Q = nx-lz$; $R = ly-mx$

W.K.T. Lagrange's Subsidary eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Choose x, y, z as multipliers of each of above fun's, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(mz-ny) + y(nx-lz) + z(ly-mx)} = 0$$

$$\frac{x\,dx+y\,dy+z\,dz}{m\cancel{xz}-\cancel{nxy}+\cancel{nxy}-\cancel{lzy}+\cancel{lzy}-\cancel{mxz}}=0$$

$$x\,dx+y\,dy+z\,dz=0$$

Integrating on both sides

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$x^2+y^2+z^2=c_1 \quad -(2)$$

choose l,m,n as multipliers of each of above fun's, we get

$$\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=0$$

$$\frac{l\,dx+m\,dy+n\,dz}{\cancel{lmz}-\cancel{lny}+\cancel{mnx}-\cancel{mlz}+\cancel{nly}-\cancel{mnx}}=0$$

$$l\,dx+m\,dy+n\,dz=0$$

Integrating on both sides

$$lx+my+nz=c_2 \quad -(3)$$

$\therefore$ (2) and (3) are req. sol^n's of given eq^n.

iv) Solve $(y-zx)p+(x+yz)q=x^2+y^2$

Given eq^n is $(y-zx)p+(x+yz)q=x^2+y^2 \quad -(1)$

compare (1) with $Pp+Qq=R$

where $P=y-zx$; $Q=x+yz$; $R=x^2+y^2$

W.K.T. integral eq^n of (1) is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{y-zx}=\frac{dy}{x+yz}=\frac{dz}{x^2+y^2}$$

choose $x,-y,z$ as multipliers of each of above fun's we get

$$\frac{x\,dx-y\,dy+z\,dz}{x(y-zx)-y(x+yz)+z(x^2+y^2)}=0$$

$$\frac{x\,dx-y\,dy+z\,dz}{\cancel{xy}-\cancel{zx^2}-\cancel{yx}-\cancel{y^2z}+\cancel{zx^2}+\cancel{zy^2}}=0$$

$$x\,dx-y\,dy+z\,dz=0$$

Integrating on both sides

$$\frac{x^2}{2}-\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$x^2 - y^2 + z^2 = c_1 \quad \text{—(2)}$$

choose $-x, y, -z$ as multipliers of each of above fun's, we get

$$\frac{-xdx + ydy - zdz}{-x(y-zx) + y(x+yz) - z(x^2+y^2)} = 0$$

$$\frac{-xdx + ydy - zdz}{-xy + zx^2 + yx + y^2z - zx^2 - zy^2} = 0$$

$$-xdx + ydy - zdz = 0$$

Integrating on both sides

$$-\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}$$

$$-x^2 + y^2 - z^2 = c_2 \quad \text{—(3)}$$

∴ (2) and (3) are req. sol^n's of given eq^n.

Another soln $y, x, -1$

$$\frac{ydx + xdy - dz}{y(y-zx) + x(x+yz) - x^2 - y^2} = 0$$

$$\Rightarrow \frac{ydx + xdy - dz}{\cancel{y^2} - \cancel{xyz} + \cancel{x^2} + \cancel{xyz} - \cancel{x^2} - \cancel{y^2}} = 0$$

$\Rightarrow ydx + xdy - dz = 0$

$\Rightarrow d(xy) - dz = 0$

$xy - z = c_2$

5. Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Given eq^n is $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$ —(1)

compare (1) with $Pp + Qq = R$

where $P = x(y^2+z)$; $Q = -y(x^2+z)$; $R = z(x^2-y^2)$

W.K.T. Lagrange's subsidary eq^n of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

choose $x, y, -1$ as multipliers of each of above functions, we get

$$\frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = 0$$

$$\frac{xdx + ydy - dz}{\cancel{x^2y^2} + \cancel{zx^2} - \cancel{x^2y^2} - \cancel{zy^2} - \cancel{zx^2} + \cancel{zy^2}} = 0$$

$$xdx + ydy - dz = 0$$

Integrate on both sides

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$x^2 + y^2 - 2z = c_1 \quad \text{—(2)}$$

choose $1/x, 1/y, 1/z$ as multipliers of each of above functions, we g

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{\cancel{x}}\cancel{x}(y^2+z) + \frac{1}{\cancel{y}}(-\cancel{y})(x^2+z) + \frac{1}{\cancel{z}}\cancel{z}(x^2-y^2)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + \cancel{z} - x^2 - \cancel{z} + \cancel{xy} - \cancel{y^2}} = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrate on both sides

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$xyz = c_2 \quad —(3)$$

$\therefore$ (2) and (3) are req. sol^n's of given equation.

6, Solve $(x+2z)p + (4zx - y)q = 2x^2 + y$

Given equation is $(x+2z)p + (4zx-y)q = 2x^2+y$ —(1)

Compare (1) with $Pp + Qq = R$

where $P = (x+2z)$; $Q = 4zx - y$; $R = 2x^2+y$

W.K.T. Lagrange's subsidary eq^n of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(x+2z)} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y}$$

choose $-2x, 1, 1$ as multipliers of each of above functions, we

get $$\frac{-2x\,dx + \cancel{y}\,dy + \cancel{z}\,dz}{-2x(x+2z) + 1(4zx-y) + 1(2x^2+y)} = 0$$

$$\frac{-2x\,dx + \cancel{y}\,dy + \cancel{z}\,dz}{\cancel{-2x^2} - \cancel{4xz} + \cancel{4zx} - \cancel{y} + \cancel{2x^2} + \cancel{y}} = 0$$

$-2x\,dx + \cancel{y}\,dy + \cancel{z}\,dz = 0$ $\quad \Rightarrow -2\left(\frac{x^2}{2}\right) + y + z = c_1$

Integrate on both sides $\quad \Rightarrow -x^2 + y + z = c_1$

$\cancel{\left[-2\left(\frac{x^2}{2}\right) + \left(\frac{y^2}{2}\right) + \left(\frac{z^2}{2}\right) = \frac{c_1}{2}\right.}$ $\quad \Rightarrow x^2 - y - z = c_1$

$\cancel{-2x^2 + y^2 + z^2 = c_1 \quad —(2)\Big]}$ ×

$$-\cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + y + z = c_1$$

$$x^2 - y - z = +c_1 \quad —(2)$$

choose $+y, +x, -2z$ as multipliers of each of above functions, we get

$$\frac{+y\,dx + x\,dy - (2z\,dz)}{y(x+2z) + x(4zx-y) - 2z(2x^2+y)} = 0$$

$$\frac{y\,dx + x\,dy - 2z\,dz}{\cancel{xy} + \cancel{2zy} + \cancel{4zx^2} - \cancel{xy} - \cancel{4zx^2} - \cancel{2zy}} = 0$$

$$y\,dx + x\,dy - 2z\,dz = 0$$

Integrate on both sides

$$\int d(xy) - \int 2z\,dz = \int 0$$

$$xy - z^2 = c_2 \quad -(3)$$

$\therefore$ (2) and (3) are req. solutions of above eqn.

Method:- III

1, Solve $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

Given equation is of the form Lagrange's auxilary eqn

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad -(1)$$

choose 1, 1, 1 as multipliers of each of functions, we get

$$\frac{dx + dy + dz}{\cancel{xy} - \cancel{zx} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{zy}} = 0$$

$$dx + dy + dz = 0$$

Integrate on both sides

$$x + y + z = c_1 \quad -(2)$$

Choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers of each of above functions, we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{\cancel{x}}\cancel{x}(y-z) + \frac{1}{\cancel{y}}\cancel{y}(z-x) + \frac{1}{\cancel{z}}\cancel{z}(x-y)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}} = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrate on both sides

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$$

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2 \quad -(3)$$

$\therefore$ (2) and (3) are required soln's of given equation.

2, Solve $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$

Given equation is $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$ — (1)

choose x, y, z as multipliers of each of above functions, we get

$$\frac{ax\,dx + by\,dy + cz\,dz}{(b-c)xyz + xyz(c-a) + xzy(a-b)} = 0$$

$$\frac{ax\,dx + by\,dy + cz\,dz}{\cancel{bxyz} - \cancel{cxyz} + \cancel{cxyz} - \cancel{axyz} + \cancel{axyz} - \cancel{bxyz}} = 0$$

$$ax\,dx + by\,dy + cz\,dz = 0$$

Integrate on both sides

$$\int ax\,dx + \int by\,dy + \int cz\,dz = \int 0$$

$$a\left(\frac{x^2}{2}\right) + b\left(\frac{y^2}{2}\right) + c\left(\frac{z^2}{2}\right) = \frac{c_1}{2}$$

$$ax^2 + by^2 + cz^2 = c_1 \quad \text{—(2)}$$

choose $\frac{1}{yz}, \frac{1}{zx}, \frac{1}{xy}$ as multipliers of each of above functions we get

$$\frac{a\,dx\frac{1}{yz} + \frac{1}{zx}b\,dy + c\,dz\frac{1}{xy}}{\frac{1}{\cancel{yz}}\cancel{yz}(b-c) + \cancel{zx}\frac{1}{\cancel{zx}}(c-a) + \frac{1}{\cancel{xy}}\cancel{xy}(a-b)} = 0$$

$$\frac{\frac{a}{yz}dx + \frac{b}{zx}dy + \frac{c}{xy}dz}{\cancel{b} - \cancel{c} + \cancel{c} - \cancel{a} + \cancel{a} - \cancel{b}} = 0$$

$$\frac{ax\,dx + by\,dy + cz\,dz}{xyz} = 0$$

Integrate on both sides

$$a\left(\frac{x^2}{2}\right) + b\left(\frac{y^2}{2}\right) + c\left(\frac{z^2}{2}\right) = \frac{c_2}{2}$$

$$ax^2 + by^2 + cz^2 = c_2 \quad \text{—(3)}$$

$\therefore$ (2) and (3) are req. sol^n's of given equation.

3, Solve $\frac{dx}{xz-y} = \frac{dy}{yz-x} = \frac{dz}{1-z^2}$

Given eq^n is $\frac{dx}{xz-y} = \frac{dy}{yz-x} = \frac{dz}{1-z^2}$ —(1)

choose $z, 1, x$ as multipliers of each of above functions, we get

$$\frac{z\,dx + dy + x\,dz}{z(xz-y)+yz-x+x-xz^2} = 0$$

$$\frac{z\,dx + dy + x\,dz}{\cancel{xz^2} - \cancel{zy} + \cancel{yz} - \cancel{x} + \cancel{x} - \cancel{xz^2}} = 0$$

Integrate on both sides

$$\int d(xz) + \int dy = \int 0$$

$$xz + y = c_1 \quad -(2)$$

choose 1, z, y as multipliers of each of above functions, we get

$$\frac{dx + z\,dy + y\,dz}{\cancel{xz} - \cancel{y} + \cancel{zy^2} - \cancel{xz} + \cancel{y} - \cancel{z^2y}} = 0$$

$$dx + z\,dy + y\,dz = 0$$

Integrate on both sides

$$\int dx + \int d(zy) = \int 0$$

$$x + zy = c_2 \quad -(3)$$

$\therefore$ (2) and (3) are req. soln's of given equation.

4, Solve $\dfrac{dx}{x^2(y^3-z^3)} = \dfrac{dy}{y^2(z^3-x^3)} = \dfrac{dz}{z^2(x^3-y^3)}$

Given eqn is $\dfrac{dx}{x^2(y^3-z^3)} = \dfrac{dy}{y^2(z^3-x^3)} = \dfrac{dz}{z^2(x^3-y^3)} \quad -(1)$

choose x, y, z as multipliers of each of above functions, we get

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x^3y^3} - \cancel{x^3z^3} + \cancel{y^3z^3} - \cancel{x^3y^3} + \cancel{z^3x^3} - \cancel{z^3y^3}} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

Integrate on both sides

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 + z^2 = c_1 \quad -(2)$$

choose $1/x^2$, $1/y^2$, $1/z^2$ as multipliers of each of above functions, we

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{\cancel{x^2}}\cancel{x^2}(y^3-z^3) + \frac{1}{\cancel{y^2}}\cancel{y^2}(z^3-x^3) + \cancel{z^2}(x^3-y^3)\frac{1}{\cancel{z^2}}} = 0$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrate on both sides

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -c_2$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2 \quad -(3)$$

$\therefore$ (2) and (3) are req. soln's of given equation.

9. Solve $x^2p + y^2q = nxy$

Given eqⁿ is $x^2p+y^2q = nxy$ —(1)

compare (1) with $Pp+Qq=R$

where $P = x^2$; $Q = \cancel{q}y^2$; $R = nxy$

W.K.T. Lagrange's ~~arbitrary~~ auxil eqⁿ of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y\cancel{q}^2} = \frac{dz}{nxy} \quad —(2)$$

Take first and second functions of eqⁿ (2)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$x^{-2}dx = y^{-2}dy$$

Integrate on both sides

$$-\frac{1}{x} + \frac{1}{y} = -c_1 \text{ where } c_1 \text{ is arbitrary constant} \longrightarrow (3)$$

$$-y + x = -c_1xy$$

$$x - y + c_1xy = 0$$

$$y - x - c_1xy = 0$$

$$\frac{y-x}{xy} = c_1$$

choose $\frac{1}{x}, -\frac{1}{y}, \frac{c_1}{n}$ as multipliers of each of above functions, we get

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz}{\frac{1}{x}(x^2) - \frac{1}{y}(y^2) + \frac{c_1}{n}(nxy)} = 0$$

Substitude c_1 value in above eqⁿ

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz}{\cancel{x} - \cancel{y} + \left(\frac{\cancel{y-x}}{n\cancel{xy}}\right)(\cancel{n}\cancel{x}y)} = 0$$

$$\frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0$$

Integrate on both sides

$$\log x - \log y + \frac{c_1}{n}\cancel{\log} z = \log c_2$$

$$\log\left(\frac{x}{y}\right) + \cancel{\log}(z)\frac{c_1}{n} = \log c_2$$

$$\log\left(\frac{x}{y}\right) + \cancel{\log}\left(\frac{y-x}{xy}\right)\left(\frac{z}{n}\right) = \log c_2 \quad —(4)$$

∴ (3) and (4) are req. soln's of given eqⁿ.

(or)

$$\phi\left(\frac{y-x}{xy}, \log\left(\frac{x}{y}\right) + \left(\frac{y-x}{xy}\right)\left(\frac{z}{n}\right)\right) = 0$$

5. Find the integral curves of the equation $\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$

Given equation is $\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$ —— (1)

Taking second and third inequalities of eq^n (1)

$$\frac{dy}{y} = \frac{dz}{z+y^2}$$

$$\Rightarrow \frac{dz}{dy} = \frac{z+y^2}{y}$$

$$\Rightarrow \frac{dz}{dy} = \frac{z}{y} + y \Rightarrow \frac{dz}{dy} - \frac{z}{y} = y \quad —— (2)$$

It is a linear differential equation.

This equation can be written as

$$\frac{d}{dy}\left(\frac{z}{y}\right) = 1$$

$$\because \frac{d}{dy}\left(\frac{z}{y}\right) = 1$$

$$\frac{y\frac{dz}{dy} - z}{y^2} = 1$$

$$y\frac{dz}{dy} - z = y^2$$

$$\frac{dz}{dy} - \frac{z}{y} = y$$

Integrate on both sides

$$\frac{z}{y} = y + c_1 \quad (3)$$

$$z = y^2 + c_1 y \quad —— (3)$$

Taking first and second functions

$$\frac{dx}{x+z} = \frac{dy}{y}$$

$$\frac{dx}{dy} = \frac{x+z}{y} = \frac{x}{y} + \frac{z}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + (y + c_1) \quad (\because \text{from } (3))$$

If we regard $\frac{dx}{dy}$ 'y' as the independent variable and 'x' as the dependent variable in above equation and we write it in the form

$$\frac{d}{dy}\frac{x}{y} = \frac{c_1}{y} + 1$$

Integrate on both sides

$$\int \frac{d}{dy}\frac{x}{y} = \int \frac{c_1}{y}\,dy + 1$$

$$\frac{x}{y} = c_1 \log y + y + c_2$$

$$x = c_1(\log y)y + y^2 + c_2 y$$

$$x = \left(\frac{z}{y} - y\right)(\log y)y + y^2 + c_2 y$$

$$x = \left(\frac{z-y^2}{\cancel{y}}\right)(\log y)\cancel{y} + y^2 + c_2 y$$

$$x = (z-y^2)(\log y) + y^2 + c_2 y \quad —— (4)$$

$\therefore$ (3) and (4) are req. sol^n's of given equation.

6, $\dfrac{dx}{z^2-2yz-y^2} = \dfrac{dy}{x(y+z)} = \dfrac{dz}{x(y-z)}$

Given equation :- $\dfrac{dx}{z^2-2yz-y^2} = \dfrac{dy}{x(y+z)} = \dfrac{dz}{x(y-z)}$ —(1)

Taking second and third functions

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\Rightarrow \frac{dy}{y+z} = \frac{dz}{y-z}$$

$\Rightarrow (y-z)dy = (y+z)dz$

$\Rightarrow y\,dz + z\,dz = y\,dy - z\,dy$

$\Rightarrow y\,dz + z\,dy = y\,dy - z\,dz$

$\Rightarrow -d(yz) + y\,dy - z\,dz = 0$

Integrating on both sides

$$\Rightarrow \frac{y^2}{2} - \frac{z^2}{2} - yz = \frac{c_1}{2}$$

$\Rightarrow y^2 - z^2 - 2yz = c_1$ —(2)

choose x, y, z as multipliers of each of functions of (1), we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(z^2-2yz-y^2) + yx(y+z) + xz(y-z)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{x\cancel{z^2} - 2y\cancel{xz} - \cancel{y^2x} + \cancel{xy^2} + \cancel{xyz} + \cancel{xzy} - \cancel{xz^2}} = 0$$

$x\,dx + y\,dy + z\,dz = 0$

Integrating on both sides

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$x^2 + y^2 + z^2 = c_2$ —(3)

$\therefore$ (2) and (3) are req. soln's of given eqn.

7, Solve $(3x+y-z)p + (x+y-z)q = 2(z-y)$

Given equation is $(3x+y-z)p + (x+y-z)q = 2(z-y)$ —(1)

Compare (1) with $Pp + Qq = R$

where $P = 3x+y-z$; $Q = x+y-z$; $R = 2(z-y)$

W.K.T. Lagrange's subsidary eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(3x+y-z)} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}$$

choose $1, -3, -1$ as multipliers of each of above functions

$$\frac{dx - 3dy - dz}{3\cancel{x}+\cancel{y}-\cancel{z}-3\cancel{x}-3\cancel{y}+3\cancel{z}-2\cancel{z}+2\cancel{y}} = 0$$

$dx - 3dy - dz = 0$ Integrating on both sides.

$\cancel{dx - 3dy}$ $x - 3y - z = c_1$ —(2)

$z = x - 3y - c_1$

To find another soln, we consider first two fractions and substitu-de above 'z' value.

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z}$$

$$\frac{dx}{3x+y-x+3y+c_1} = \frac{dy}{x+y-x+3y+c_1}$$

$$\frac{dx}{2x+4y+c_1} = \frac{dy}{4y+c_1}$$

Let $u = 4y + c_1$

$du = 4dy \Rightarrow dy = \frac{du}{4}$

$$\frac{dx}{2x+u} = \frac{du}{4u}$$

$$\frac{dx}{du} = \frac{2x+u}{4u} = \frac{\cancel{2}x}{\cancel{4}_2 u} + \frac{\cancel{u}}{4\cancel{u}}$$

$$\frac{dx}{du} - \frac{x}{2u} = \frac{1}{4} \quad —(3)$$

which is a linear partial diff. eqn, to solve this we use integrating factor

$I.F = e^{\int P(x)dx}$ where $P = -\frac{1}{2u}$

$= e^{\int \frac{-1}{2u} du}$

$= e^{-1/2 \int \frac{1}{u} du}$

$= e^{-1/2 \log u} = u^{-1/2} = \frac{1}{+\sqrt{u}}$

∴ Solution of eqn (3) is

$x(I.F) = \int Q\, du\, (I.F) + c$

$x\left(\frac{1}{\sqrt{u}}\right) = \int \frac{1}{4} \cdot \frac{1}{\sqrt{u}} du + c_2$

$\frac{x}{\sqrt{u}} = \frac{1}{4}\int u^{-1/2} du + c_2$

$= \frac{1}{4} \frac{u^{1/2}}{1/2} + c_2$

$= \frac{1}{4} 2u^{1/2} + c_2$

$\frac{x}{\sqrt{u}} = \frac{\sqrt{u}}{2} + c_2 \Rightarrow x = \left(\frac{\sqrt{u}}{2} + c_2\right)\sqrt{u}$

$x = \frac{u}{2} + c_2\sqrt{u}$

$$x = \left(\frac{4y+c_1}{2}\right) + c_2\sqrt{4y+c_1}$$

$$x = \frac{4y+x-3y-z}{2} + c_2\sqrt{4y+x-3y-z}$$

$$x = \frac{x+y-z}{2} + c_2\sqrt{x+y-z}$$

$$x - \frac{(x+y-z)}{2} = c_2\sqrt{x+y-z}$$

$$\frac{2x-(x+y-z)}{2} = c_2\sqrt{x+y-z}$$

$$\frac{x-y+z}{2\sqrt{x+y-z}} = c_2 \Rightarrow \frac{x-y+z}{\sqrt{x+y-z}} = c_2 \quad -(3)$$

$\therefore$ (2) and (3) are required soln's of given eqn.

8, Solve $(y^3x-2x^4)p+(2y^4-x^3y)q = 9z(x^3-y^3)$

Given eqn is $(y^3x-2x^4)p+(2y^4-x^3y)q = 9z(x^3-y^3) \quad -(1)$

compare (1) with $Pp+Qq=R$

where $P = y^3x-2x^4$; $Q = 2y^4-x^3y$; $R = 9z(x^3-y^3)$

W.K.T. Lagrange's subsidary eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z(x^3-y^3)} \quad -(2)$$

Taking first and second fractions of eqn (2)

$$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y}$$

$$(2y^4-x^3y)dx = (y^3x-2x^4)dy$$

Divide with x^3y^3 on both sides

$$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right)dx = \left(\frac{1}{x^2}-\frac{2x}{y^3}\right)dy$$

$$\frac{2y}{x^3}dx - \frac{1}{y^2}dx = \frac{1}{x^2}dy - \frac{2x}{y^3}dy$$

$$\frac{1}{x^2}dy - \frac{2y}{x^3}dx = \frac{2x}{y^3}dy - \frac{1}{y^2}dx$$

$$\left(\frac{1}{x^2}dy - \left(\frac{2y}{x^3}\right)dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$$

$$\left[\because d\left(\frac{y}{x^2}\right) = \frac{x^2dy - y(2x)dx}{(x^2)^2}\right.$$

$$\left. = \frac{x^2dy - 2xydx}{x^4} = \frac{1}{x^2}dy - \frac{2y}{x^3}dx\right]$$

$\Rightarrow d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$

Integrate on both sides

$\frac{y}{x^2} + \frac{x}{y^2} = c_1$ —(3) where c_1 is arbitrary constant

Choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{3z}$ as multipliers of each of above functions, we g

$$\frac{1/x\,dx + 1/y\,dy + \frac{1}{3z}dz}{\frac{1}{\cancel{x}}\cancel{x}(y^3-2x^3) + \frac{1}{\cancel{y}}\cancel{y}(2y^3-x^3) + \frac{1}{\cancel{3z}}\,\overset{3}{\cancel{9z}}(x^3-y^3)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz}{\cancel{y^3} - \cancel{2x^3} + \cancel{2y^3} - \cancel{x^3} + \cancel{3x^3} - \cancel{3y^3}} = 0$$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz = 0$

Integrate on both sides

$\log x + \log y + \frac{1}{3}\log z = \log c_2$

$\log(xyz^{1/3}) = \log c_2$

$xyz^{1/3} = c_2$ —(4) where c_2 is an arbitrary constant

$\therefore$ (3) and (4) req. soln's of given equation.

(8) $\phi\left(\frac{y}{x^2} + \frac{x}{y^2}, xyz^{1/3}\right) = 0$ where 'ϕ' is arbitrary (constant) function

10, Solve $(x-y)p + (x+y)q = 2xz$

Given equation is $(x-y)p + (x+y)q = 2xz$ —(1)

Compare (1) with $Pp + Qq = R$

where $P = x-y$; $Q = x+y$; $R = 2xz$

W.K.T Lagrange's subsidary eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \quad \text{—(2)}$$

Taking first and second functions of eqn (2)

$$\frac{dx}{x-y} = \frac{dy}{x+y}$$

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{\cancel{x}(1+\frac{y}{x})}{\cancel{x}(1-y/x)}$$

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} \quad \text{—(3)}$$

Let us take $v = \frac{y}{x}$

$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—(4)}$$

$(3) \Rightarrow \dfrac{dy}{dx} = \dfrac{1+v}{1-v}$ — (5)

$(4) \Rightarrow v + x\dfrac{dv}{dx} = \dfrac{1+v}{1-v}$

$x\dfrac{dv}{dx} = \dfrac{1+v}{1-v} - v$

$= \dfrac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$

$= \dfrac{1+v^2}{1-v}$

$\dfrac{1-v}{1+v^2}dv = \dfrac{dx}{x}$

$\Rightarrow \dfrac{1}{1+v^2}dv - \dfrac{v}{1+v^2}dv = \dfrac{1}{x}dx$

multiply with '2' on both sides $\left[\because \int \dfrac{f'(x)}{f(x)}dx = \log f(x)\right]$

$\Rightarrow \dfrac{2}{1+v^2}dv - \dfrac{2v}{1+v^2}dv = \dfrac{2}{x}dx$

$\Rightarrow 2\tan^{-1}v - \log \overset{(1+v^2)}{f(v)} = 2\log x - \log c_1)$

$\Rightarrow 2\tan^{-1}v - \log(1+v^2) = \log x^2 - \log c_1$

$\Rightarrow 2\tan^{-1}v = \log\left(\dfrac{x^2}{c_1}\right) + \log(1+v^2)$

$\Rightarrow 2\tan^{-1}v = \log\left(\dfrac{x^2}{c_1}(1+v^2)\right)$

$\Rightarrow \dfrac{x^2}{c_1}(1+v^2) = e^{2\tan^{-1}v}$

substitude $v = y/x$

$\Rightarrow x^2\left(1+\dfrac{y^2}{x^2}\right) = c_1 e^{2\tan^{-1}\left(\frac{y}{x}\right)}$

$\Rightarrow (x^2+y^2) = c_1 e^{2\tan^{-1}\left(\frac{y}{x}\right)}$

$\Rightarrow c_1 = (x^2+y^2) e^{-2\tan^{-1}\left(\frac{y}{x}\right)}$ — (6)

choose $1, 1, -\frac{1}{z}$ as multipliers of each of fractions in (2), we get

$\dfrac{dx+dy-\frac{1}{z}dz}{x-\cancel{y}+x+\cancel{y}-\frac{1}{\cancel{z}}(2x\cancel{z})} = 0$

$dx+dy-\frac{1}{z}dz = 0$

Integrate on both sides

$x+y-\log z = \log c_2$

$\Rightarrow x+y = \log(zc_2)$

$\Rightarrow zc_2 = e^{x+y}$

$\Rightarrow c_2 = \dfrac{e^{x+y}}{z}$ — (7)

$\therefore$ (6) and (7) are required soln's of given equation.

(8) $\phi\left((x^2+y^2)e^{-2\tan^{-1}\left(\frac{y}{x}\right)}, \dfrac{e^{x+y}}{z}\right) = 0$ is the arbitrary function.

$$x\frac{dv}{dx} = -v\left[\frac{4+4v^2}{1+3v^2}\right]$$

$$= -4v\left(\frac{1+v^2}{1+3v^2}\right)$$

$$\frac{1+3v^2}{v(1+v^2)}dv = -4\frac{dx}{x}$$

$$\frac{1+3v^2}{v(1+v^2)}dv + 4\frac{dx}{x} = 0 \quad -(4)$$

To simplify $\frac{1+3v^2}{v(1+v^2)}$ we use partial fractions

$$\frac{1+3v^2}{v(1+v^2)} = \frac{A}{v} + \frac{(Bv+C)}{1+v^2} \quad -(5)$$

$$\frac{1+3v^2}{\cancel{v(1+v^2)}} = \frac{A(1+v^2)+(Bv+C)v}{\cancel{(1+v^2)v}}$$

$$1+3v^2 = A(1+v^2)+(Bv+C)v$$

Take $v=0$

$$\boxed{1 = A}$$

Substitude $A=1$ and take $v=1$ in (5)

$$\frac{1+3}{1(1+1)} = 1 + \frac{(B+C)}{2}$$

$$\frac{4}{\cancel{2}} = \frac{2+(B+C)}{\cancel{2}} \Rightarrow B+C=2 \quad -(6)$$

Substitude $A=1$ and take $v=-1$ in (5)

$$\frac{1+3}{-1(1+1)} = -1 + \frac{(-B+C)}{1+1}$$

$$\frac{4}{-2} = -1+\frac{(-B+C)}{2} \Rightarrow -\frac{4}{\cancel{2}} = \frac{-2+(-B+C)}{\cancel{2}}$$

$$\Rightarrow -B+C = -2 \quad -(7)$$

Solve (6) and (7)

$$\cancel{B}+C=2$$
$$\cancel{-B}+C=-2$$
$$2C=0 \Rightarrow C=0 \text{ then } B=2$$

Substitude A, B, C in eqn (5)

$$\frac{1+3v^2}{v(1+v^2)} = \frac{1}{v} + \frac{(2v)}{1+v^2}$$

$$(5) \Rightarrow \left(\frac{1}{v}+\frac{2v}{1+v^2}\right)dv + 4\frac{dx}{x} = 0$$

Integrate on both sides $\quad \left(\because \int\frac{f'(x)}{f(x)}dx = \log f(x)\right)$

$$\log v + \log(1+v^2) + 4\log x = \log c_2$$

$$\log(v(1+v^2)x^4) = \log c_2$$

$\Rightarrow v(1+v^2)x^4 = c_2$

$\Rightarrow \left(\frac{y}{x}\right)\left(1+\frac{y^2}{x^2}\right)x^4 = c_2$

$\Rightarrow \left(\frac{y}{x}\right)\left(\frac{x^2+y^2}{x^2}\right)x^4 = c_2$

$\Rightarrow \frac{y(x^2+y^2)x^{\cancel{4}}}{x\cancel{3}} = c_2$

$\Rightarrow xy(x^2+y^2) = c_2$ — (B)

$\therefore$ A and B are req. sol^n's of given equation.

12; Solve $x^2q + y^2p = x^2y^2z^2$

Given eq^n is $y^2p + x^2q = x^2y^2z^2$ — (1)

Compare (1) with $Pp + Qq = R$

where $P = y^2$; $Q = x^2$; $R = x^2y^2z^2$

W.K.T. Lagrange's auxilary equation of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$$

Taking first and second fractions

$\frac{dx}{y^2} = \frac{dy}{x^2} \Rightarrow x^2dx = y^2dy$

Integrate on both sides

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$

$\Rightarrow x^3 - y^3 = c_1$ — (2)

choose $x^2, y^2, -2/z^2$ as multipliers of each of above functions, we g

$$\frac{x^2dx + y^2dy - \frac{2}{z^2}dz}{x^2y^2 + x^2y^2 - \frac{2}{\cancel{z^2}}(x^2y^2\cancel{z^2})} = 0$$

$x^2dx + y^2dy - 2z^{-2}dz = 0$

Integrate on both sides

$\frac{x^3}{3} + \frac{y^3}{3} - 2\left(-\frac{1}{z}\right) = \log c_2$

$\frac{z(x^3+y^3)+6}{3z} = \log c_2$ — (3)

$\therefore$ (3) and (4) are req. soln's of given equation.

$\therefore \phi\left(x^3 - y^3, \frac{z(x^3+y^3)+6}{3z}\right) = 0$

13 Solve the equations $\frac{dx}{y+\alpha z} = \frac{dy}{z+\beta x} = \frac{dz}{x+\gamma y}$

Given equation is $\frac{dx}{y+\alpha z} = \frac{dy}{z+\beta x} = \frac{dz}{x+\gamma y}$ — (1)

W.K.T. Lagrange's auxilary eq^n of (1) is

Let us take each of these ratios is equal to

$$\frac{\lambda dx + \mu dy + \gamma dz}{\lambda(y+\alpha z) + \mu(z+\beta x) + \gamma(x+\gamma y)}$$

If λ, μ & γ are constant multipliers, this equation will be an exact differential equation if it is of the form

$$\frac{1}{\rho} \frac{\lambda dx + \mu dy + \gamma dz}{\lambda x + \mu y + \gamma z}$$

This eq^n holds only if
$$\left.\begin{aligned} -\rho\lambda + \beta\mu + \gamma &= 0 \\ \lambda - \rho\mu + \gamma v &= 0 \\ \alpha\lambda + \mu - \rho v &= 0 \end{aligned}\right\} \text{—(2)}$$

Regarded as equations in λ, μ and v these equations possess a solution only if 'ρ' is a root of the equation.

$$\begin{vmatrix} -\rho & \beta & 1 \\ 1 & -\rho & \gamma \\ \alpha & 1 & -\rho \end{vmatrix} = 0 \text{ — (3)}$$

$\Rightarrow -\rho(\rho^2-\gamma) - \beta(-\rho-\alpha\gamma) + 1(1+\alpha\rho) = 0$

$\Rightarrow -\rho^3 + \gamma\rho + \beta\rho + \beta\alpha\gamma + 1 + \alpha\rho = 0$

$\Rightarrow -\rho^3 + (\alpha+\beta+\gamma)\rho + 1 + \alpha\beta\gamma = 0$ —(4)

This eq^n has three roots, we denote these roots by ρ_1, ρ_2, ρ_3

If we substitute the value of ρ_1 for ρ in eq^n (4) and substitute $\lambda = \lambda_1$, $\mu = \mu_1$ & $\gamma = \gamma_1$ then from eq^n (3) we have

$$dw' = \frac{1}{\rho_1}\left(\frac{\lambda_1 dx + \mu_1 dy + \gamma_1 dz}{\lambda_1 x + \mu_1 y + \gamma_1 z}\right)$$

Integrate on both sides

$w' = \frac{1}{\rho_1}\log(\lambda_1 x + \mu_1 y + \gamma_1 z) + \log c_1$

$\Rightarrow w' = \log(\lambda_1 x + \mu_1 y + \gamma_1 z)^{1/\rho_1} \cdot c_1$

$lly\ w'' = \log(\lambda_2 x + \mu_2 y + \gamma_2 z)^{1/\rho_2} \cdot c_2$

$\rho_1, c_1 \rho_2$
$\rho_1 - c_2\, \rho_3$

using this, we can write eq^n (3) as

$(\lambda_1 x + \mu_1 y + \gamma_1 z)^{1/\rho_1} = c_1(\lambda_2 x + \mu_2 y + \gamma_2 z)^{1/\rho_2}$, where c_1 is constant

In a similar way, we have

$(\lambda_3 x + \mu_3 y + \gamma_3 z)^{1/\rho_3} \cdot c_2 = (\lambda_1 x + \mu_1 y + \gamma_1 z)^{1/\rho_2}$, where c_2 is constant

Sec- Orthogonal Trajectories of a System of curves on a surface

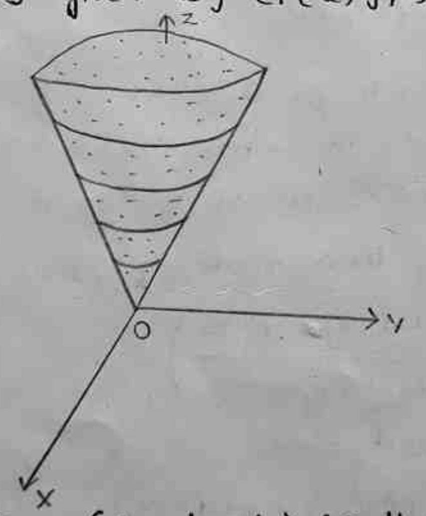
The problem of finding the orthogonal trajectories of a sys-tem of plane curves is well-known. In three-dimension, corresponding problem is:

Given a surface $F(x,y,z)=0$ —(1) and a system of curves on it, to find a system of curves each of which lies on the surface ① and cuts every curve of the given system at right angles.

The new system of curves is called the "system of orthogonal trajectories on the surface of the given system of curves". The original system of curves may be thought of the intersection of the surface ① with one parameter family of surfaces $G(x,y,z)$ $=c_1$ —(2)

Ex:- For example, consider a system of circles (shown by full lines in the figure) is form on the cone $x^2+y^2 = z^2\tan^2\alpha$ —(1)

Sol:- Let us consider the surface eqⁿ $f(x,y,z)=0$ —(2)
Then the orthogonal trajectories on the surface of the given system of curves is given by $G(x,y,z)=c_1$ —(3)



The tangential direction (dx,dy,dz) to the given curve at the point (x,y,z) on the surface ② satisfies the eqⁿ

$F(x,y,z) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$ —(4) &

$G(x,y,z) = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz = 0$ —(5) [Here $c_1 = 0$]

Here the triods (dx,dy,dz) satisfies the eqⁿ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —(6)

where $P = \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}$ } Taking (y,z)

$$Q = \frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}$$ Taking (z,x)

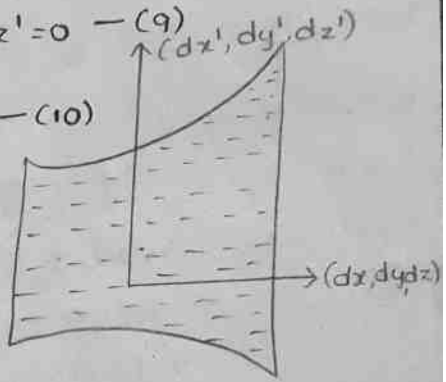
$$R = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}$$ Taking (x,y) — (7)

$\therefore$ The curve through the point (x,y,z) of the orthogonal system has the tangential direction (dx', dy', dz') (shown in below figure) (which lies on the surface (2) to)

$$\frac{\partial F}{\partial x}dx' + \frac{\partial F}{\partial y}dy' + \frac{\partial F}{\partial z}dz' = 0 \quad \text{— (8)}$$

and is $\perp$lar to the original system of curves

$\therefore$ from eq^n (6) we have $P\,dx' + Q\,dy' + R\,dz' = 0$ — (9)



From (8) and (9) we have $\frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'}$ — (10)

To get P', Q' and R' we use eq^n (7)

In eq^n (7) substitude in P, $\frac{\partial G}{\partial z} = R$ & $\frac{\partial G}{\partial y} = Q$

$\Rightarrow P' = R\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial z}$

In eq^n (7) substitude in Q, $\frac{\partial G}{\partial x} = P$ & $\frac{\partial G}{\partial z} = R$

$\Rightarrow Q' = P\frac{\partial F}{\partial z} - R\frac{\partial F}{\partial x}$

In eq^n (7) substitude in R, $\frac{\partial G}{\partial y} = Q$; $\frac{\partial G}{\partial x} = P$

$\Rightarrow R' = Q\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial y}$

$\therefore \frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'}$

where
$$\left.\begin{aligned} P' &= R\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial z} \\ Q' &= P\frac{\partial F}{\partial z} - R\frac{\partial F}{\partial x} \\ R' &= Q\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial y} \end{aligned}\right\} \text{— (11)}$$

$\therefore$ The above solutions with the relation $F(x,y,z) = 0$ gives the orthogonal trajectories.

$\rightarrow$ This can be illustrated by doing another example.

Ex:- Find the orthogonal trajectories on the cone $x^2+y^2 = z^2\tan^2\alpha$ of its intersections with the family of planes llel to $z=0$

Given equation of the cone is $x^2+y^2 = z^2\tan^2\alpha$ — (1)

Given condition is that the cone intersect, the family of planes llel to $z=0$

Thus, the given system of circles on the cone is characterized by the pair of equations.

$x\,dx + y\,dy = \tan^2\alpha\, z\,dz,\ z = 0$ — (2)

Substitude $z=0$

(2) $\Rightarrow x\,dx + y\,dy = \tan^2\alpha \cdot 0$

$x\,dx + y\,dy = 0$

Compare above eq^n with $Pp + Qq = R$

where $P = x$; $Q = y$; $R = 0$

We know that Lagrange's auxilary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$$

Take $\frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow y\,dx - x\,dy = 0$ —(3)

∴ The system of orthogonal trajectories is determined by the pair o

equations. $x\,dx + y\,dy = z\tan^2\alpha\,dz$; $y\,dx - x\,dy = 0$

Take L.H.S. of above eq^n

$x\,dx + y\,dy$

After Integration we get, $x^2 + y^2$

Divide RHS with x^2+y^2 while writing the Lagrange's auxilary eq^n, w

$$\frac{dx}{x} = \frac{dy}{y} = \frac{z\tan^2\alpha}{x^2+y^2}\,dz \quad \text{—(4)}$$

Take first and second functions

$$\frac{dx}{x} = \frac{dy}{y}$$

$\Rightarrow x = y c_1$ —(5)

Pfaffin Differential Forms and Equations

Pfaffin diff-- eqn :- The expression $\sum_{i=1}^{n} F_i(x_1, x_2, x_3 \cdots x_n)\,dx$ —(1) in which F_i, $i=1,2,3 \cdots n$ are functions of some or all of the 'n' independent variables $(x_1, x_2, x_3 \cdots x_n)$ is called a "pfaffin diff-- form".

The relation $\sum_{i=1}^{n} F_i\,dx_i = 0$ is called a "pfaffin diff-- equation".

Note:- A Pfaffin differ--eqn is integrable whenever $\bar{x}\cdot\text{curl}\,\bar{x} = 0$ where $\bar{x} = (P, Q, R)$ and $\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

Theorem:1 A pfaffin diff-- eqn in two variables always possess an integrating factor.

Let x, y be two variables of a pfaffin differential eqn of the form $P\,dx + Q\,dy = 0$ —(1)

$\Rightarrow P\,dx = -Q\,dy$

$\Rightarrow \frac{dy}{dx} = -\frac{P}{Q}$

Let $\frac{dy}{dx} = -\frac{P}{Q} = f(x,y)$ —(2)

Since P & Q are single valued functions

$\Rightarrow \frac{dy}{dx} = \frac{-P}{Q} = f(x,y)$ is also a single valued function.

$\Rightarrow \exists$ excatly one parametri family of function 'ϕ' $\ni \phi(x,y) = c$ —(3)

Now differentiating eqn (3) we get $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$ —(4)

From eqn's (1) & (4) we have $P = \frac{\partial\phi}{\partial x}$; $Q = \frac{\partial\phi}{\partial y}$

$$\Rightarrow \frac{1}{P}\frac{\partial\phi}{\partial x} = \frac{1}{Q}\frac{\partial\phi}{\partial y} = \mu \text{ (say)}$$

$$\Rightarrow \frac{1}{P}\frac{\partial\phi}{\partial x} = \mu \text{ \& } \frac{1}{Q}\frac{\partial\phi}{\partial y} = \mu$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = \mu P \text{ \& } \frac{\partial\phi}{\partial y} = \mu Q \quad \text{—(5)}$$

Now multiply eqn (1) with μ

(1) $\Rightarrow \mu P\,dx + \mu Q\,dy = 0$

$\Rightarrow \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$ ($\because$ from (5))

From eqn's (3) and (4) we have, $d(\phi(x,y)) = 0$

$\therefore$ 'μ' is the integrating factor of given pfaffin differential eqn in two variables.

Hence proved.

1. Show that the pfaffin differential equation $y\,dx + x\,dy + 2z\,dz = 0$ is integrable and find its integral.

Given eqn is:- $y\,dx + x\,dy + 2z\,dz = 0$ — (1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = y$, $Q = x$; $R = 2z$

Since $\bar{x} = (P, Q, R)$

$\Rightarrow \bar{x} = (y, x, 2z)$

$$\text{curl } \bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & 2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(2z) - \frac{\partial x}{\partial z}\right] - j\left[\frac{\partial}{\partial x}(2z) - \frac{\partial y}{\partial z}\right] + k\left[\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right]$$

$= i(0-0) - j(0-0) + k(1-1)$

$= 0$

Now $\bar{x} \cdot \text{curl } \bar{x} = (y, x, 2z) \cdot (0)$

$= 0$

$\therefore \bar{x} \cdot \text{curl } \bar{x} = 0$

$\therefore$ The given pfaffin differential equation is integrable.

To find the integrals.

$y\,dx + x\,dy + 2z\,dz = 0$

$d(xy) + 2z\,dz = 0$

Integrate on both sides

$xy + \cancel{2}\left(\frac{z^2}{\cancel{2}}\right) = c_1$

$xy + z^2 = c_1$

Since $u(x,y,z) = c_1 \Rightarrow u(x,y,z) = xy + z^2$ is the req. integral of give

2, Show that the pfaffin differential eq^n is $yz\,dx + 2xz\,dy - 3xy\,d$

is integrable and find its integral.

Given eq^n is :- $yz\,dx + 2xz\,dy - 3xy\,dz = 0$ — (1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = yz$; $Q = 2xz$; $R = -3xy$

Since $\bar{x} = (P, Q, R)$

$= (yz, 2xz, -3xy)$

$$\text{Now curl } \bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & -3xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-3xy) - \frac{\partial}{\partial z}(2xz)\right] - j\left[\frac{\partial}{\partial x}(-3xy) - \frac{\partial}{\partial z}(yz)\right] + k\left[\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial y}(yz)\right]$$

$= i[-3x - 2x] - j[-3y - y] + k[2z - z]$

$= -5xi + 4jy + zk$

Now $\bar{x} \cdot \text{curl } \bar{x} = (yz, 2xz, -3xy)(-5x, 4y, z)$

$= -5xyz + 8xyz - 3xyz$

$= 0$

$\therefore \bar{x} \cdot \text{curl } \bar{x} = 0$

$\therefore$ The given pfaffin differentiable eq^n is integrable.

To find the integrals.

$$yz\,dx + 2xz\,dy - 3xy\,dz = 0$$

Divide with xyz

$$\frac{dx}{x} + 2\frac{dy}{y} - 3\frac{dz}{z} = 0$$

Integrate on both sides

$$\log x + \log y^2 - \log z^3 = \log c_2$$

$$\log\left(\frac{xy^2}{z^3}\right) = \log c_2$$

$$\frac{xy^2}{z^3} = c_2 \qquad \text{Since } u(x,y,z) = c_2$$

$u(x,y,z) = \dfrac{xy^2}{z^3}$ is the req. integral of given eq^n.

3. Show that the pfaffin diff... eq^n $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$ is integrable and find its integral.

Given eq^n is :- $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2-xy)\,dz = 0$ —(1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = y^2+yz$; $Q = xz+z^2$; $Q = y^2-xy$

Since $\bar{x} = (P, Q, R)$

$$= (y^2+yz,\ xz+z^2,\ y^2-xy)$$

Now $\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+yz & xz+z^2 & y^2-xy \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(y^2-xy) - \frac{\partial}{\partial z}(xz+z^2)\right] - j\left[\frac{\partial}{\partial x}(y^2-xy) - \frac{\partial}{\partial z}(y^2+yz)\right] + k\left[\frac{\partial}{\partial x}(xz+z^2) - \frac{\partial}{\partial y}(y^2+yz)\right]$$

$$= i(2y - 2x - 2z) - j(-y-y) + k(\cancel{z} - 2y - \cancel{z})$$

$$= (-2x+2y-2z)(i) - j(-2y) + k(-2y)$$

$$= (-2x+2y-2z)i + 2yj - 2yk$$

Now $\bar{x}\cdot\operatorname{curl}\bar{x} = (-2x+2y-2z,\ 2y,\ -2y)\,(y^2+yz,\ xz+z^2,\ y^2-xy)$

$$= \cancel{-2xy^2} + \cancel{2y^3} - \cancel{2zy^2} - \cancel{2xyz} + \cancel{2y^2z} - \cancel{2yz^2} + \cancel{2xyz} + \cancel{2yz^2} + \cancel{2xy^2} - \cancel{2y^3}$$

$$= 0$$

$\therefore \bar{x}\cdot\operatorname{curl}\bar{x} = 0$

$\therefore$ The given pfaffin differentiable eq^n is integrable.

To find the integrals.

$$(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$$

for simplication, treat 'z' as a constant $\Rightarrow dz = 0$

$$\therefore y(y+z)dx + z(x+z)dy = 0$$

Divide with $y(y+z)(x+z)$

$$\frac{dx}{x+z} + \frac{z\,dy}{y(y+z)} = 0 \quad —(2)$$

from eq^n (2), consider $\frac{z}{y(y+z)}$

Here we use partial fractions.

$$\frac{z}{y(y+z)} = \frac{A}{y} + \frac{B}{y+z} \quad —(A)$$

$$\frac{z}{y(\cancel{y+z})} = \frac{A(y+z)+By}{y(\cancel{y+z})}$$

$$z = A(y+z) + By \quad —(3)$$

Put $z = -y$

$$-y = +By$$

$$\boxed{B = -1}$$

Substitude 'B' value in (3) and put $z = y$

$$z = A(2z) - y$$

$$y = 2Az - z$$

$$2z = 2Az$$

$$\boxed{A = 1}$$

$$\therefore (A) \Rightarrow \frac{z}{y(y+z)} = \frac{1}{y} - \frac{1}{y+z} \quad —(4)$$

Substitude (4) in (2) $\frac{dx}{x+z} + \left(\frac{1}{y} - \frac{1}{y+z}\right)dy = 0$

Integrate on both sides

$$\int \frac{1}{x+z}\,dx + \int \frac{1}{y}\,dy - \int \frac{1}{y+z}\,dy = 0$$

$$\log(x+z) + \log y - \log(y+z) = \log c$$

$$\log\left(\frac{(x+z)y}{y+z}\right) = \log c$$

$$c = \frac{(x+z)y}{y+z}$$

we know that $u(x,y,z) = c$

$u(x,y,z) = \frac{(x+z)y}{y+z}$ is the req. integral ~~sol^n~~ of given equation

4, $yz\,dx + xz\,dy + xy\,dz = 0$

Given eq^n is $yz\,dx + xz\,dy + xy\,dz = 0 \quad —(1)$

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = yz$; $Q = xz$; $R = xy$

Since $\bar{x} = (P, Q, R)$

$$= (yz, xz, xy)$$

$$\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$$

$= i\left(\frac{\partial}{\partial y} xy - \frac{\partial}{\partial z} xz\right) - j\left(\frac{\partial}{\partial x} xy - \frac{\partial}{\partial z} yz\right) + k\left(\frac{\partial}{\partial x} xz - \frac{\partial}{\partial y} yz\right)$

$= i(x - x) - j(y - y) + k(z - z)$

$= i(0) - j(0) + k(0)$

Now $\bar{x} \cdot \operatorname{curl}\bar{x} = (yz, xz, xy)(0,0,0)$

$= yz(0) + xz(0) + xy(0)$

$= 0$

$\therefore$ The given pfaffin differentiable eq^n is integrable.

To find the integrals.

$yz\,dx + xz\,dy + xy\,dz = 0$ (or) $d(xyz) = 0$.

Divide with xyz $\qquad xyz = c_1$

$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrate on both sides

$\log x + \log y + \log z = \log c_1$

$xyz = c_1$

Since $u(x, y, z) = c_1$

$\Rightarrow u(x, y, z) = xyz$ is the required integral of given equation.

5, $x\,dy - y\,dx - 2x^2\,dz = 0$

Given eq^n is $x\,dy - y\,dx - 2x^2\,dz = 0$ — (1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = -y$; $Q = x$; $R = -2x^2$

Since $\bar{x} = (P, Q, R) = (-y, x, -2x^2)$

$$\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & -2x^2 \end{vmatrix}$$

$= \left[i\left(\frac{\partial}{\partial y}(-2x^2) - \frac{\partial}{\partial z}(x)\right] - j\left(\frac{\partial}{\partial x}(-2x^2) - \frac{\partial}{\partial z}(-y)\right) + k\left(\frac{\partial x}{\partial x} - \frac{\partial}{\partial y}(y)\right)\right.$

$= i(0-0) - j(-4x+0) + k(1-1)$

$= 4xj$

Now $\bar{x} \cdot \operatorname{curl}\bar{x} = (-y, x, -2x^2)(4x)$

$= -y(0) + x(4x) + 0(-2x^2)$

$= 4x^2$

$\therefore$ The given pfaffin differential eq^n is integrable.

To find the integrals

$xdy - ydx - 2x^2dz = 0$

Divide with x^2

$-\frac{y}{x^2}dx + \frac{1}{x}dy - 2dz = 0$

Integrate on both sides

$-y\left(\int \frac{1}{x^2}dx\right) + \int \frac{1}{x}dy - 2\int dz = \int 0$

$-y\left(-\frac{1}{x}\right) + \frac{y}{x} - 2z = c$

$\frac{y}{x} + \frac{y}{x} - 2z = c$

$2\left(\frac{y}{x} - z\right) = c$

Since $u(x,y,z) = c$

$\therefore u(x,y,z) = 2\left(\frac{y}{x} - z\right)$ is the required integral of given equation.

6. $z(z+y)dx + z(z+x)dy - 2xy\,dz = 0$

Given eqn :- $z(z+y)dx + z(z+x)dy - 2xy\,dz = 0$ — (1)

compare (1) with $Pdx + Qdy + Rdz = 0$

where $P = z(z+y)$; $Q = z(z+x)$; $R = -2xy$

Since $\bar{x} = (P, Q, R)$

$= (z(z+y), z(z+x), 2xy)$

Now $\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z(z+y) & z(z+x) & -2xy \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}(-2xy) - \frac{\partial}{\partial z}(z(z+x))\right] - j\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial z}(z(z+y))\right] + k\left[\frac{\partial}{\partial x} z(z+x) - \frac{\partial}{\partial y} z(z+y)\right]$

$= i(-2x - 2z - x) - j(-2y - 2z - y) + k(z - z)$

$= i(-3x - 2z) - j(-3y - 2z) + k(0)$

Now $\bar{x}\cdot\text{curl}\,\bar{x} = (z^2+zy, z^2+zx, 2xy)(-3x-2z, +3y+2z, 0)$

$= (z^2+zy)(-3x-2z) + (z^2+zx)(3y+2z) + 2xy(0)$

$= -3xz^2 - \cancel{2z^3} - \cancel{3zxy} - 2z^2y + 3z^2y + \cancel{2z^3} + \cancel{3xyz} + 2z^2x$

$= -xz^2 + z^2y$

$\therefore \bar{x}\cdot\text{curl}\,\bar{x} \neq 0$

$\therefore$ The given pfaffin differential equation is not integrable because

$\bar{x}\cdot\text{curl}\,\bar{x} \neq 0$

7, $2xz\,dx + z\,dy - dz = 0$

Given equation :- $2xz\,dx + z\,dy - dz = 0$ —— (1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = 2xz$; $Q = z$; $R = -1$

Since $\bar{x} = (P, Q, R)$

$= (2xz, z, -1)$

Now curl $\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz & z & -1 \end{vmatrix}$

$= i\left(\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(z)\right) - j\left(\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}(2xz)\right) + k\left(\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial y}(2xz)\right)$

$= i(0-1) - j(0-2x) + k(0-0)$

$= i(-1) + j(2x) + k(0)$

Therefore, $\bar{x} \cdot \text{curl}\,\bar{x} = (2xz, z, -1)(-1, 2x, 0)$

$= -\cancel{2x}z + \cancel{2}xz + 0$

$= 0$

$\therefore$ The given pfaffin differential equation is integrable.

To find the integrals.

$2xz\,dx + z\,dy - dz = 0$

Divide with $\frac{1}{z}$ z

$2x\,dx + dy - \frac{1}{z}dz = 0$

Integrate on both sides

$\cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + y - \log z = c$

$x^2 + y - \log z = c$

we know that $u(x, y, z) = c$

$\therefore u(x, y, z) = x^2 + y - \log z$ is the required integral of given eq^n.

8. $(6x+yz)\,dx + (xz-2y)\,dy + (xy+2z)\,dz = 0$

Given equation is $(6x+yz)dx + (xz-2y)dy + (xy+2z)dz = 0$ —(1)

Compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = 6x+yz$; $Q = xz-2y$; $R = xy+2z$

Since $\bar{x} = (P, Q, R)$

$= (6x+yz,\ xz-2y,\ xy+2z)$

$$\text{curl}\ \bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (6x+yz) & (xz-2y) & (xy+2z) \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(xy+2z) - \frac{\partial}{\partial z}(xz-2y)\right] - j\left[\frac{\partial}{\partial x}(xy+2z) - \frac{\partial}{\partial z}(6x+yz)\right] + k\left[\frac{\partial}{\partial x}(x\right.$ $\left. - \frac{\partial}{\partial y}(6x+yz)\right]$

$= i(x-x) - j(y-y) + k(z-z)$

$= i(0) - j(0) + k(0)$

$= 0$ $\bar{x}\cdot\text{curl}\,\bar{x} = (6x+yz,\ xz-2y,\ xy+2z)(0,0,0)$

now $\bar{x}$ curl $\bar{x} = (6x+4z)(0) + (xz-2y)(0), (xy+2z)(0)$

$= 0$

$\therefore \bar{x}$ curl $\bar{x} = 0$

$\therefore$ The given pfaffin differential equation is integrable.

To find the integrals.

$(6x+yz)dx + (xz-2y)dy + (xy+2z)dz = 0$

$6x\,dx + yz\,dx + xz\,dy - 2y\,dy + xy\,dz + 2z\,dz = 0$

$6x\,dx + 2z\,dz - 2y\,dy + yz\,dx + xz\,dy + xy\,dz = 0$

Integrate on both sides

$6\left(\frac{x^2}{2}\right) + \cancel{2}\left(\frac{z^2}{\cancel{2}}\right) - \cancel{2}\left(\frac{y^2}{\cancel{2}}\right) + (xyz) = c.$

$3x^2 - y^2 + z^2 + xyz = c$

Since $u(x,y,z) = c$

$u(x,y,z) = 3x^2 - y^2 + z^2 + xyz$

is the required integral of given equation.

9. $(y^2+xz)\,dx + (x^2+yz)\,dy + 3z^2\,dz = 0$

Given equation is $(y^2+xz)dx + (x^2+yz)dy + 3z^2dz = 0$ —(1)

compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = (y^2+xz)$; $Q = (x^2+yz)$; $R = (3z^2)$

Since $\bar{x} = (P, Q, R)$

$= (y^2+xz,\ x^2+yz,\ 3z^2)$

$$\text{curl }\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ y^2+xz & x^2+yz & 3z^2 \end{vmatrix}$$

$$= i\left(\frac{d}{dy}(3z^2) - \frac{d}{dz}(x^2+yz)\right] - j\left[\frac{d}{dx}3z^2 - \frac{d}{dz}(y^2+xz)\right] + k\left[\frac{d}{dx}(x^2+yz) - \frac{d}{dy}(y^2+$$

$$= i(-y) - j(-x) + k(2x-2y)$$

Now $\bar{x}\cdot\text{curl }\bar{x} = (-y, +x, 2x-2y)(y^2+xz, x^2+yz, 3z^2)$

$$= -y(y^2+xz) + x(x^2+yz) + (2x-2y)(3z^2)$$

$$= -y^3 + x^3 - \cancel{xyz} + \cancel{xyz} + 6xz^2 - 6yz^2$$

$$= x^3 - y^3 + 6(xz^2 - yz^2) \neq 0 \quad \therefore \bar{x}\cdot\text{curl }\bar{x} \neq 0$$

$\therefore$ The given pfaffin differentiablequation is not integrable.

10. $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$

Given eqn is :- $(x^2z-y^3)dx + 3xy^2dy + x^3dz = 0$ —(1)

compare (1) with $Pdx + Qdy + Rdz = 0$

where $P = x^2z - y^3$; $Q = 3xy^2$; $R = x^3$

since $\bar{x} = (P, Q, R)$

$$= ((x^2z - y^3), 3xy^2, x^3)$$

$$\text{curl }\bar{x} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ x^2z-y^3 & 3xy^2 & x^3 \end{vmatrix}$$

$$= i\left[\frac{d}{dy}(x^3) - \frac{d}{dz}(3xy^2)\right] - j\left[\frac{d}{dx}(x^3) - \frac{d}{dz}(x^2z-y^3)\right] + k\left[\frac{d}{dx}(3xy^2) - \frac{d}{dy}(x^2z-y^3)\right]$$

$$= i(0-0) - j(3x^2 - x^2) + k(3y^2 + 3y^2)$$

$$= -j(2x^2) + k(6y^2)$$

Now $\bar{x}\cdot\text{curl }\bar{x} = (x^2z - y^3, 3xy^2, x^3)(0, -2x^2, 6y^2)$

$$= (x^2z - y^3)(0) - (3xy^2)(2x^2) + x^3(6y^2)$$

$$= -6x^3y^2 + 6x^3y^2 = 0$$

$\therefore \bar{x}\cdot\text{curl }\bar{x} = 0$

$\therefore$ The given pfaffin differential equation is integrable.

To find the integrals.

$(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$

$x^2(zdx + xdz) + y^2(3xdy - ydx) = 0$

Divide with x^2

$$zdx + xdz = +\frac{y^3}{x^2}dx - \frac{3y^2}{x}dy$$

$$\Rightarrow d(xz) + \left(\frac{3y^2}{x}dy - \frac{y^3}{x^2}dx\right) = 0$$

$$d(xz) + \left(\frac{3xy^2dy - y^3dx}{x^2}\right) = 0$$

$$d(xz) + d\left(\frac{y^3}{x}\right) = 0$$

Integrate on both sides

$$xz + \frac{y^3}{x} = C$$

$$\frac{x^2z + y^3}{x} = C$$

Since $u(x,y,z) = c$

$u(x,y,z) = \left(\frac{x^2z+y^3}{x}\right)$ is the required integral of given equatio

II, $(yz+xyz)dx + (zx+xyz)dy + (xy+xyz)dz = 0$

Given equation is $(yz+xyz)dx + (zx+xyz)dy + (xy+xyz)dz = 0$ —(1)

Compare (1) with $Pdx + Qdy + Rdz = 0$

where $P = yz+xyz$; $Q = zx+xyz$; $R = xy+xyz$

Since $\bar{x} = (P, Q, R) = (yz+xyz, zx+xyz, xy+xyz)$

$$\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+xyz & zx+xyz & xy+xyz \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(xy+xyz) - \frac{\partial}{\partial z}(zx+xyz)\right] - j\left[\frac{\partial}{\partial x}(xy+xyz) - \frac{\partial}{\partial z}(yz+xyz)\right] + k\left[\frac{\partial}{\partial x}(zx+ \ldots) - \frac{\partial}{\partial y}(yz+xyz)\right]$

$= i(\cancel{x} + xz - \cancel{x} - xy) - j(\cancel{y} + yz - \cancel{y} - xy) + k(\cancel{z} + yz - \cancel{z} - xz)$

$= i(xz - xy) - j(yz - xy) + k(yz - zx)$

Now $\bar{x}\cdot\text{curl}\,\bar{x} = (yz+xyz, zx+xyz, xy+xyz)(xz-xy, xy-yz, yz-\ldots)$

$= (yz+xyz)(xz-xy) + (zx+xyz)(xy-yz) + (xy+xyz)(yz-\ldots)$

$= \cancel{xyz^2} - \cancel{xy^2z} + \cancel{x^2yz^2} - \cancel{x^2y^2z} + \cancel{x^2yz} + \cancel{x^2y^2z} - \cancel{xyz^2} - \ldots$

$+ \cancel{xy^2z} - \cancel{x^2yz} + \cancel{xy^2z^2} - \cancel{x^2yz^2}$

$= 0$

$\therefore \bar{x}\cdot\text{curl}\,\bar{x} = 0$

$\therefore$ The given differential equation is integrable.

To find the integrals.

$(yz+xyz)dx + (zx+xyz)dy + (xy+xyz)dz = 0$

$(yz\,dx + zx\,dy + xy\,dz) + (xyz\,dx + xyz\,dy + xyz\,dz) = 0$

~~$d(xyz) + xyz(dx+dy+dz) = 0$~~ ~~$d(xyz) + xyz(dx+dy+dz) = 0$~~

~~Integrate on both sides.~~ Divide with xyz ~~$d(\cdot) + dx + dy + dz$~~

~~$x + y + \cdot$~~ ~~$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$~~ ~~$xyz + dx + dy + dz = 0$~~

Divide with xyz on both sides

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz + (dx+dy+dz) = 0$$

Integrate on both sides.

$$\log x + \log y + \log z + x+y+z = \log c$$

$$\log(xyz) + x+y+z = \log c$$

Since $u(x,y,z) = c$

$u(x,y,z) = \log(xyz)+x+y+z$ is the req. integral of given equation.

12. $(yz+2x)dx + (zx-2z)dy + (xy-2y)dz = 0$

Given equation is $(yz+2x)dx + (zx-2z)dy + (xy-2y)dz = 0$ — (1)

compare (1) with $Pdx + Q\,dy + Rdz = 0$

where $P = yz+2x$; $Q = zx-2z$; $R = xy-2y$.

Now $\bar{x} = (P,Q,R) = (yz+2x,\ zx-2z,\ xy-2y)$

$$\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+2x & zx-2z & xy-2y \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(xy-2y) - \frac{\partial}{\partial z}(zx-2z)\right) - j\left(\frac{\partial}{\partial x}(xy-2y) - \frac{\partial}{\partial z}(yz+2x)\right) + k\left(\frac{\partial}{\partial x}(zx-2z) - \frac{\partial}{\partial y}(yz+2x)\right)$$

$$= i(\cancel{x}-2-\cancel{x}+2) - j(y-y) + k(z-z)$$

$$= i(0) - j(0) + k(0) = 0$$

Now $\bar{x}\cdot\text{curl}\,\bar{x} = (yz+2x,\ zx-2z,\ xy-2y)\,(0,0,0)$

$$= (yz+2x)(0) + (zx-2z)(0) + (xy-2y)(0)$$

$$= 0$$

$\therefore \bar{x}\cdot\text{curl}\,\bar{x} = 0$

$\therefore$ The given pfaffin differential equation is integrable.

To find the integrals.

$$(yz+2x)\,dx + (zx-2z)dy + (xy-2y)dz = 0$$

$$yzdx + 2xdx + zxdy - 2zdy + xydz - 2ydz = 0$$

$$xzdy + yzdx + xydz + 2(x\,dx) - 2(zdy+ydz) = 0$$

$$d(xyz) + 2(xdx) - 2(zdy+ydz) = 0$$

Integrate on both sides.

$$xyz + \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) - 2(yz) = c$$

$$xyz + x^2 - 2yz = c$$

Since $u(x,y,z) = c$

$$u(x,y,z) = (xyz + x^2 - 2yz)$$

is the required integral of given equation.

13. Find whether the following equations are integrable or not ?

1. $(2x+y^2+2xz)\,dx + 2xy\,dy + x^2\,dz = 0$

Given equation:- $(2x+y^2+2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ —(1)

Compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = 2x+y^2+2xz$; $Q = 2xy$; $R = x^2$

We know that $\bar{x} = (P, Q, R) = (2x+y^2+2xz,\ 2xy,\ x^2)$

Now $\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+y^2+2xz & 2xy & x^2 \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(2xy)\right] - j\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(2x+y^2+2xz)\right] + k\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(2x+y^2 \ldots\right.$

$= i(0-0) - j(2x-2x) + k(2y-2y)$

$= i(0) - j(0) + k(0)$

Therefore, $\bar{x}\cdot\operatorname{curl}\bar{x} = (2x+y^2+2xz, 2xy, x^2)(0,0,0)$

$= (2x+y^2+2xz)(0) + 2xz(0) + 2xy(0)$

$= 0 + 0 + 0 = 0$

$\therefore \bar{x}\cdot\operatorname{curl}\bar{x} = 0$

$\therefore$ The given differential equation is integrable

2. $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2+xy)\,dz = 0$

Given equation:- $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2+xy)\,dz = 0$ —(1)

Compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

where $P = y^2+yz$; $Q = xz+z^2$; $R = y^2+xy$

We know that $\bar{x} = (P, Q, R)$

$= (y^2+yz,\ xz+z^2,\ y^2+xy)$

Now $\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+yz & xz+z^2 & y^2+xy \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}(y^2+xy) - \frac{\partial}{\partial z}(xz+z^2)\right] - j\left[\frac{\partial}{\partial x}(y^2+xy) - \frac{\partial}{\partial z}(y^2+yz)\right] + k\left[\frac{\partial}{\partial x}(xz+z^2) - \frac{\partial}{\partial y}(\ldots\right.$

$= i(2y + \cancel{x} - \cancel{x} - 2z) - j(\cancel{x} - y) + k(\cancel{z} - 2y - \cancel{z}) = i(2y-2z) - j(0) + k(-2y)$

Now $\bar{x}\cdot\operatorname{curl}\bar{x} = (y^2+yz, xz+z^2, y^2+xy)(2y-2z, 0, -2y)$

$= (2y-2z)(y^2+yz) + (xz+z^2)(0) + (y^2+xy)(-2y)$

$= 2y^3 - 2zy^2 + 2y^2z - 2yz^2 - 2y^3 - 2xy^2$

$= -2xy^2 - 2yz^2$

$\neq 0$

$\therefore \bar{x}\cdot\operatorname{curl}\bar{x} \neq 0$

$\therefore$ The given differential eqn is not integrable.

3. $(zdx+zdy)+2(x+y+\sin z)dz=0$

Given eq^n is $(zdx+zdy)+2(x+y+\sin z)dz=0$ —(1)

Compare (1) with $Pdx+Qdy+Rdz=0$

where $P=z$; $Q=z$; $R=2x+2y+2\sin z$

we know that $\bar{x}=(P,Q,R)=(z,z,2x+2y+2\sin z)$

$$\text{Now curl }\bar{x}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & z & 2x+2y+2\sin z \end{vmatrix}$$

$$=i\left[\frac{\partial}{\partial y}(2x+2y+2\sin z)-\frac{\partial}{\partial z}(z)\right]-j\left[\frac{\partial}{\partial x}(2x+2y+2\sin z)-\frac{\partial}{\partial z}(z)\right]+k\left[\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial y}(z)\right]$$

$=i[2-1]-j(2-1)+k(0-0)$

$=i-j$

Now, $\bar{x}\cdot\text{curl}\,\bar{x}=(z,z,2x+2y+2\sin z)(1,-1,0)$

$=z-z$

$=0$

$\therefore$ The given differentiable eq^n is integrable.

Theorem:-2 A necessary and sufficient condition that $\exists$ a relation between two functions $u(x,y)$ and $v(x,y)$ i.e., $F(u,v)=0$, not involving 'x' and 'y' explicitly that $\frac{\partial(u,v)}{\partial(x,y)}=0$

Proof:- Given that $u(x,y)$ and $v(x,y)$ be two functions of 'x' & 'y'.

$$0=F(u,v)\Leftrightarrow\frac{\partial(u,v)}{\partial(x,y)}=0$$

Part i:- First suppose that $F(u,v)=0$ —(1) is a relation between 'u' & 'v' not involving 'x' and 'y'

Claim:- $\frac{\partial(u,v)}{\partial(x,y)}=0$

Now differentiating eq^n (1) with respect to 'x'

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0 \quad —(2)$$

Now differentiate eq^n (1) with respect to 'y'

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=0 \quad —(3)$$

Now (2) $\times\frac{\partial v}{\partial y}$ $-$ (3) $\times\frac{\partial v}{\partial x}$ we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\cancel{\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}-\frac{\partial v}{\partial x}\cdot\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}-\cancel{\frac{\partial v}{\partial x}\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}}=0$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]=0$$

$$\Rightarrow\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=0$$

$$\Rightarrow\frac{\partial(u,v)}{\partial(x,y)}=0\quad\left[\because\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}\right]$$

Conversely, Suppose that $\frac{\partial(u,v)}{\partial(x,y)}=0$

Claim:- $F(u,v)=0$

ie, we have to prove that $F(u,v)=0$ is a relation between 'u' &'v'

not involving 'x' and 'y'

On the contrary way, suppose that $F(u,v,x)=0$ —(4)

Now differentiate (4) with respect to 'x'

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}=0$$

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial F}{\partial x}=0 \quad —(5)$$

Now differentiate (4) with respect to 'y'

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=0 \quad —(6)$$

Now, (5) $\times\frac{\partial v}{\partial y}$ − (6) $\times\frac{\partial v}{\partial x}$ we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\cancel{\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}+\frac{\partial F}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\cancel{\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}}=0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]+F_x\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left[\frac{\partial(u,v)}{\partial(x,y)}\right]+F_x\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\partial F}{\partial u}(0)+F_x\frac{\partial v}{\partial y}=0$$

$$\Rightarrow F_x\frac{\partial v}{\partial y}=0$$

$$\Rightarrow F_x=0$$

ie, $\frac{\partial F}{\partial x}=0$

Thus, the function 'F' does not involve 'x' and 'y'

Therefore, $F(u,v)=0$.

ie, $F(u,v)=0 \Leftrightarrow \frac{\partial(u,v)}{\partial(x,y)}=0$

** Hence proved.

***Theorem:-3** If 'X' is a vector such that $\bar{X}\cdot\text{curl}\,\bar{X}=0$ and 'μ' is an arbitrary function of (x,y,z) then $\mu\bar{X}\cdot\text{curl}\,\mu\bar{X}=\bar{0}$

Proof:- Given that 'X' is a vector.

so let $X=(P,Q,R)$

$\Rightarrow \bar{X}=(P\bar{i}, Q\bar{j}, R\bar{k})$

$\Rightarrow$ If μ is a arbitrary funtion of (x,y,z) then $\mu\bar{X}=(\mu P\bar{i}, \mu Q\bar{j}, \mu R\bar{k})$

Also suppose $\bar{X}\cdot\text{curl}\,\bar{X}=0$

Claim:- $\mu\bar{X}\cdot\text{curl}\,\mu\bar{X}=0$

Since $\bar{x} = (P\bar{i}, Q\bar{j}, R\bar{k})$

we know that $\operatorname{curl}\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$\operatorname{curl}\bar{x} = i\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - j\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + k\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$

Now, $\bar{x}\cdot\operatorname{curl}\bar{x} = P\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ —(1)

By assumption, $\bar{x}\cdot\operatorname{curl}\bar{x} = 0$

$\Rightarrow P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$ —(2)

Since $\mu\bar{x} = (\mu P\bar{i}, \mu Q\bar{j}, \mu R\bar{k})$

Now, $\operatorname{curl}\mu\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mu P & \mu Q & \mu R \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}(\mu R) - \frac{\partial}{\partial z}(\mu Q)\right] - j\left[\frac{\partial}{\partial x}(\mu R) - \frac{\partial}{\partial z}(\mu P)\right] + k\left[\frac{\partial}{\partial x}(\mu Q) - \frac{\partial}{\partial y}(\mu P)\right]$

$= i\left[R\frac{\partial \mu}{\partial y} + \mu\frac{\partial R}{\partial y} - Q\frac{\partial \mu}{\partial z} - \mu\frac{\partial Q}{\partial z}\right] - j\left[R\frac{\partial \mu}{\partial x} + \mu\frac{\partial R}{\partial x} - P\frac{\partial \mu}{\partial z} - \mu\frac{\partial P}{\partial z}\right] + k\left[Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x} - P\frac{\partial \mu}{\partial y} - \mu\frac{\partial P}{\partial y}\right]$

$= i\left[\mu\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] + \left(R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z}\right)\right] - j\left[\mu\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + \left(R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z}\right)\right] + k\left[\mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + \left(Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y}\right)\right]$

Now $\mu\bar{x}\cdot\operatorname{curl}\mu\bar{x} = \mu P\left[\mu\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \left(R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z}\right)\right] - \mu Q\left[\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \left(R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z}\right)\right] + \mu R\left[\mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + \left(Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y}\right)\right]$

$= \mu^2 P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \mu PR\frac{\partial \mu}{\partial y} - \mu PQ\frac{\partial \mu}{\partial z} - \mu^2 Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) - \mu QR\frac{\partial \mu}{\partial x} + \mu PQ\frac{\partial \mu}{\partial z} + \mu^2 R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + \mu RQ\frac{\partial \mu}{\partial x} - \mu PR\frac{\partial \mu}{\partial y}$

$= \mu^2\left[P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] - \mu\left[P\left(Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y}\right) + Q\left(R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z}\right) + R\left(Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y}\right)\right]$

$= \mu^2(\bar{x}\cdot\operatorname{curl}\bar{x}) - 0$ ($\because$ By (1) & Negative value is taken as negligible ie, '0')

$= \mu^2(0)$ ($\because \bar{x}\cdot\operatorname{curl}\bar{x} = 0$)

$\therefore \mu\bar{x}\cdot\operatorname{curl}\mu\bar{x} = 0$

$\therefore$ If $\bar{x}\cdot\operatorname{curl}\bar{x} = 0$ and 'μ' is an arbitrary function of (x, y, z) we have

$\mu\bar{x}\cdot\operatorname{curl}\mu\bar{x} = 0$

Hence proved.

The necessary and sufficient condition that the pfaffin differential equation $\bar{X} \cdot \bar{dr} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ integrable is that $\bar{X} \cdot \text{curl}\,\bar{X} = 0$

Proof:- Necessary Condition:-

Given that $\bar{X} \cdot \bar{dr} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ is integrable.

Claim:- $\bar{X} \cdot \text{curl}\,\bar{X} = 0$

Suppose $\bar{X} \cdot \bar{dr} = 0$

ie, $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ is integrable —(1)

Since eq^n (1) is integrable then $\exists$ arbitrary function $u(x,y,z)$ and $\mu(x,y,z) \ni du = \mu(x,y,z)\,[P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz]$

$\Rightarrow du = \mu(x,y,z)\,(P\,dx + Q\,dy + R\,dz)$

$\Rightarrow du = \mu P\,dx + \mu Q\,dy + \mu R\,dz$ —(2)

Since 'u' is a function of x, y, z

$\Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$\Rightarrow du = u_x\,dx + u_y\,dy + u_z\,dz$ —(3)

Now, equating the coefficients of dx, dy and dz we get

$$\left.\begin{aligned} u_x &= \mu P \\ u_y &= \mu Q \\ u_z &= \mu R \end{aligned}\right\} \text{—(4)}$$

Since $\bar{X} = (P, Q, R)$

$\Rightarrow \bar{X} = (P\bar{i} + Q\bar{j} + R\bar{k})$

Now $\mu\bar{X} = \mu P\bar{i} + \mu Q\bar{j} + \mu R\bar{k}$

$\mu\bar{X} = u_x\bar{i} + u_y\bar{j} + u_z\bar{k}$ ($\because$ By (4))

$\Rightarrow \mu\bar{X} = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}$

$\Rightarrow \mu\bar{X} = \nabla u$

Since we know that $\text{curl}(\nabla u) = 0$

$\Rightarrow \text{curl}\,\mu\bar{X} = 0$ $[\because \mu\bar{X} = \nabla u]$

Now, $\mu\bar{X} \cdot \text{curl}\,\mu\bar{X} = [\mu P, \mu Q, \mu R]\,(0,0,0)$

$\therefore \mu\bar{X} \cdot \text{curl}\,\mu\bar{X} = 0$

By above theorem, $\mu\bar{X} \cdot \text{curl}\,\mu\bar{X} = 0 \Leftrightarrow \bar{X} \cdot \text{curl}\,\bar{X} = 0$

$\therefore \bar{X} \cdot \text{curl}\,\bar{X} = 0$

$\therefore$ The necessary condition that the pfaffin differential equation to be integrable is that $\bar{X} \cdot \text{Curl}\,\bar{X} = 0$

Sufficient part:- Suppose $\bar{X} \cdot \text{curl}\,\bar{X} = 0$

Claim:- $\bar{X} \cdot \bar{dr} = 0$ is integrable.

ie, we have to prove that $Pdx + Qdy + Rdz = 0$ is integrable —(5)

where P, Q, R are functions of x, y, z

For simplification, Let 'z' be a constant.

$\Rightarrow dz = 0$

Substitude $dz = 0$ in eq^n (5)

(5) $\Rightarrow Pdx + Qdy = 0$ —(6)

which is a pfaffin differential equation in two variables.

By known lemma, "there always exists an integrating factor for the pfaffin differential equation in two variables".

$\Rightarrow$ By known theorem eq^n (6) is always integrable.

$\therefore$ There exists a function $u(x, y, z) = c$, which is a sol^n of eq^n (6)

where 'c' is a constant.

Since by eq^n (3) we have $du = u_x dx + u_y dy$ —(7)

clearly (6) and (7) are similar equations

$\therefore \frac{u_x}{P} = \mu$ and $\frac{u_y}{Q} = \mu$

$\Rightarrow \mu P = u_x$ and $\mu Q = u_y$ —(8)

Now multiply eq^n (5) with 'μ' we get

$\mu P dx + \mu Q dy + \mu R dz = 0$

Substitude $\mu P = u_x$; $\mu Q = u_y$ in above eq^n we get

$u_x dx + u_y dy + \mu R\, dz = 0$

Add and subtract $u_z dz$

$\Rightarrow u_x dx + u_y dy + u_z dz - u_z dz = 0 - \mu R dz$

$\Rightarrow (u_x dx + u_y dy + u_z dz) + (\mu R - u_z)\, dz = 0$

$\Rightarrow du + (\mu R - u_z)\, dz = 0$ [$\because$ By (3)]

$\Rightarrow du + k\, dz = 0$ —(9) where $k = \mu R - u_z$

Since by our assumption, $\bar{X} \cdot \text{curl}\,\bar{X} = 0$

$\Rightarrow$ By known lemma, we have $\mu\bar{X} \cdot \text{curl}\,\mu\bar{X} = 0$

since $\mu\bar{X} = (\mu P, \mu Q, \mu R)$

since $k = \mu R - u_z$

$\Rightarrow \mu R = k + u_z$

$\therefore \mu\bar{x} = (\mu P, \mu Q, k+u_z)$

$= (u_x, u_y, k+u_z)$

$\mu\bar{x} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}+k\right) \rightarrow$

$\mu\bar{x} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) + (0,0,k)$

$\Rightarrow \mu\bar{x} = \nabla u + (0,0,k)$

$$\text{curl}\,\mu\bar{x} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}+k \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}+k\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right] - j\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+k\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right] + k\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]$$

$$= i\left(\frac{\partial k}{\partial y}\right) - j\left(\frac{\partial k}{\partial x}\right) + 0$$

$$\text{curl}\,\mu\bar{x} = \left(\frac{\partial k}{\partial y}, -\frac{\partial k}{\partial x}, 0\right)$$

Now, $\mu\bar{x}\cdot \text{curl}\,\mu\bar{x} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}+k\right)\left(\frac{\partial k}{\partial y}, \frac{-\partial k}{\partial x}, 0\right)$

$= \left(\frac{\partial u}{\partial x}\frac{\partial k}{\partial y} + \frac{\partial u}{\partial y}\left(\frac{-\partial k}{\partial x}\right) + \left(\frac{\partial u}{\partial z}+k\right)0\right)$

$= \frac{\partial u}{\partial x}\frac{\partial k}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial k}{\partial x}$

$\therefore \mu\bar{x}\cdot \text{curl}\,\mu\bar{x} = \frac{\partial(u,k)}{\partial(x,y)}$

Since $\mu\bar{x}\cdot \text{curl}\,\mu\bar{x} = 0$

$\Rightarrow \frac{\partial(u,k)}{\partial(x,y)} = 0$

Thus, 'k' can be expressed as follows of 'u' and 'z' not involving 'x' and 'y'.

$\therefore$ The equation $\frac{\partial(u,k)}{\partial(x,y)} = 0$ indicates that the relation between 'u' and 'k' is independent of 'x' and 'y'.

So, from eq^n (9) we have $du + k\,dz = 0$

ie, $du + k(u,z) = 0$

This equation possess a sol^n $\phi(u,z) = c$ where 'c' is an arbitrary constant.

Now, this sol^n can be expressed in the form $u(x,y,z)=c$ by using the pfaffin expression for 'u' in terms of x, y and z.

$\therefore$ The pfaffin differential equation $Pdx+Qdy+Rdz=0$ is integrable. Hence proved.

Theorem:- Given one integrating factor of the pfaffin differential equation $x_1dx_1 + x_2dx_2 + \dots + x_ndx_n = 0$ we can find an infinity of them.

proof:- For if $\mu(x_1, x_2, x_3 \dots x_n)$ is an integrating factor of the given eq^n, there exists a function $\phi(x_1, x_2, x_3 \dots x_n)$ with the property that

$$\mu X_i = \frac{\partial \phi}{\partial x_i}, \quad i = 1, 2, 3 \ldots n \text{ —— (1)}$$

If $\Phi(\phi)$ is an arbitrary function of 'ϕ', we find that the given pf-affin differential equation may be written in the form.

$$\mu \frac{d\Phi}{d\phi}(x_1 dx_1 + x_2 dx_2 + \cdots + x_n dx_n) = 0$$

which by virtue of the relation (1) is equivalent to

$$\frac{d\Phi}{d\phi}\left(\frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3 + \cdots + \frac{\partial \phi}{\partial x_n}dx_n\right) = 0$$

Let $\frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3 + \cdots + \frac{\partial \phi}{\partial x_n}dx_n = d\phi$

ie, $\frac{d\Phi}{d\phi} d\phi = 0$

$\Rightarrow d\Phi = 0$

which the solution $\Phi(\phi) = c$

Thus, if 'μ' is an integrating factor yielding a sol^n $\phi = c$ and if Φ is an arbitrary function of ϕ, then $\mu\left(\frac{d\Phi}{d\phi}\right)$ is also an integrating factor of the given differential equation.

Since Φ is arbitrary, there are infinitely many integrating factors of this type.

To show how the theoretical argument outlined in the Proof of theorem (A)

Theorem (A) may be used to derive the sol^n of a pfaffin differential eq^n. Hence proved.

Solutions of Pfaffin Differential Equation in Three variables

We shall now consider methods to solve the pfaffin differenti-al equations in three variables.

1. By Inspection Method:- In this method first we verify the condition for integrability. By inspecting (rearranging) the terms of the given eq^n will be written in the form $\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+ \ldots dz=0$. By integration this equation we get the primitive $v(x,y,\ldots$

1. Solve the eq^n $(x^2z-y^3)dx+3xy^2dy+x^3dz=0$ and find its primitive.

given eq^n is $(x^2z-y^3)dx+3xy^2dy+x^3dz=0$ —(1)

Compare (1) with $Pdx+Qdy+Rdz=0$

where $P=x^2z-y^3$; $Q=3xy^2$; $R=x^3$

we know that $X=(P,Q,R)$

$\Rightarrow X=(x^2z-y^3, 3xy^2, x^3)$

$\Rightarrow \bar{X}=(x^2z-y^3)\bar{i}+3xy^2\bar{j}+x^3\bar{k}$

Now $\text{curl}\,\bar{X}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z-y^3 & 3xy^2 & x^3 \end{vmatrix}$

$=i\left[\frac{\partial x^3}{\partial y}-\frac{\partial}{\partial z}(3xy^2)\right]-j\left[\frac{\partial}{\partial x}(x^3)-\frac{\partial}{\partial z}(x^2z-y^3)\right]+k\left[\frac{\partial}{\partial x}(3xy^2)-\frac{\partial}{\partial y}(x^2z-y^3)\right]$

$=i(0-0)-j(3x^2-x^2)+k(3y^2+3y^2)$

$=-2jx^2+6y^2k$

Now $\bar{X}\cdot\text{curl}\,\bar{X}=((x^2z-y^3)i+3xy^2\bar{j}+x^3\bar{k})(-2jx^2+6y^2k)$

$=3xy^2(-2x^2)+x^3(6y^2)$

$=-6x^3y^2+6x^3y^2$

$=0$

$\therefore$ The given pfaffin differential equation is integrable.

By eq^n (1) $(x^2z-y^3)dx+3xy^2dy+x^3dz=0$

$\Rightarrow x^2zdx-y^3dx+3xy^2dy+x^3dz=0$

$\Rightarrow x^2(zdx+xdz)-y^3dx+3xy^2dy=0$

Divide with x^2

$\Rightarrow (zdx+xdz)-\frac{y^3}{x^2}dx+\frac{3y^2}{x}dy=0$

$\Rightarrow d(xz)+\left(\frac{-y^3dx+3xy^2dy}{x^2}\right)=0$

$\Rightarrow d(xz)+\left(\frac{3xy^2dy-y^3dx}{x^2}\right)=0$

$$d(xz)+d\left(\frac{y^3}{x}\right)=0$$

Now integrate on both sides

$$xz+\frac{y^3}{x}=c \Rightarrow x^2z+y^3=cx$$

which is the required primitive of given eq^n.

2, Solve $zydx-zxdy-y^2dz=0$

Given eq^n is $zydx-zxdy-y^2dz=0$ —(1)

compare (1) with $Pdx+Qdy+Rdz=0$

where $P=zy;\ Q=-zx;\ R=-y^2$

we know that $X=(P,Q,R)$

$$=(zy,-zx,-y^2)$$

$$\bar{x}=zy\bar{i}-zx\bar{j}-y^2\bar{k}$$

Now $\text{curl}\,\bar{x}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zy & -zx & -y^2 \end{vmatrix}$

$=i\left(\frac{\partial}{\partial y}(-y^2)-\frac{\partial}{\partial z}(-zx)\right)-j\left(\frac{\partial}{\partial x}(-y^2)-\frac{\partial}{\partial z}(zy)\right)+k\left(\frac{\partial}{\partial x}(-zx)-\frac{\partial}{\partial y}(zy)\right)$

$=i(-2y+x)-j(0-y)+k(-z-\overset{z}{\cancel{y}})$

Now $\bar{x}\cdot\text{curl}\,\bar{x}=(zy\,i-zx\bar{j}-y^2\bar{k})\,((-2y+x)i+jy+k(\cancel{-2z}))$

$$=zy(-2y+x)-zx(y)-y^2(-z-\overset{z}{\cancel{y}})$$

$$=-2\cancel{z}y^2+\cancel{xy}z-\cancel{xy}z+\cancel{y^2}z+y\cancel{2}z=0$$

The given pfaffin differential eq^n is integrable.

$zydx-zxdy-y^2dz=0$

Divide with zy^2

$\frac{1}{y}dx-\frac{x}{y^2}dy-\frac{1}{z}dz=0$

$\left(\frac{ydx-xdy}{y^2}\right)-\frac{1}{z}dz=0 \quad d\left(\frac{x}{y}\right)-\frac{1}{z}dz=0$

Integrate on both sides

$\frac{x}{y}\ \cancel{\log z}=c \quad \frac{x}{y}-\log z=\log c$

$$\frac{x}{y}=\log(cz)$$

ie, $cz=e^{x/y}$ is the req. sol^n of given eq^n.

3,

i, **Variable Separable method:-** In this method it is possible to write the pfaffin differential eqⁿ in the form $P(x)dx + Q(y)dy + R(z)dz = 0$ then the integral surfaces are given by the eqⁿ $\int P(x)dx + \int Q(y)dy + \int R(z)dz = c$ where 'c' is a constant

1, Solve the eqⁿ $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$

Given eqⁿ is $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$ —(1)

Divide with $x^2y^2z^2$

$$\frac{a^2}{x^2}dx + \frac{b^2}{y^2}dy + \frac{c^2}{z^2}dz = 0$$

Integrate on both sides

$$a^2\left(-\frac{1}{x}\right) + b^2\left(-\frac{1}{y}\right) + c^2\left(-\frac{1}{z}\right) = c$$

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = c_1$$ where c_1 is constant

2, Solve $yzdx + xzdy + xydz = 0$

Given eqⁿ is $yzdx + xzdy + xydz = 0$ —(1)

Divide with xyz

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrate on both sides

$$\log xyz = \log c$$

$xyz = c$ where 'c' is constant

ii, **One variable Separable Method:-** In this method one variable is separable z (say) Then the given eqⁿ is of the form

$P(x,y)dx + Q(x,y)dy + R(z)dz = 0$ —(1) Then $\bar{x} = (P, Q, R)$

$$\bar{x} = (P\bar{i}(x,y) + Q\bar{j}(x,y) + \bar{k}R(z))$$

$$\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & R(z) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(R(z)) - \frac{\partial}{\partial z}Q(x,y)\right] - j\left[\frac{\partial}{\partial x}R(z) - \frac{\partial}{\partial z}P(x,y)\right] + k\left(\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right)$$

$$\text{curl}\,\bar{x} = i(0) - j(0) + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

we know that the condition for integrability is $\bar{x}\cdot\text{curl}\,\bar{x} = 0$

$$(P(x,y), Q(x,y), R(z))\left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

$$P\cdot 0 + Q\cdot 0 + R(z)\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

$\Rightarrow R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 0$

$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \quad \therefore \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

In other words, we say that $Pdx + Qdy$ is an exact diff... eq^n

$\therefore Pdx + Qdy = 0$ is exact

Let $Pdx + Qdy = du$ — (2)

Substitude (2) in (1) then

$du + R(z)\,dz = 0$

By integrating this eq^n we get the primitive of this eq^n

$\int du + R(z)\,dz = 0$

$u(x,y) + \int R(z)\,dz = C$

∴ verify the eq^n $x(y^2-a^2)dx + y(x^2-z^2)dy - z(y^2-a^2)dz = 0$ is integ-rable and solve it

Sol:- Given eq^n is $x(y^2-a^2)dx + y(x^2-z^2)dy - z(y^2-a^2)dz = 0$ — (1)

$X = (P, Q, R) = (x(y^2-a^2), y(x^2-z^2), -z(y^2-a^2))$

compare (1) with $Pdx + Qdy + Rdz = 0$

$\bar{X} = (x\bar{i}(y^2-a^2) + y\bar{j}(x^2-z^2) + -z\bar{k}(y^2-a^2))$

$$\text{curl}\,\bar{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(y^2-a^2) & y(x^2-z^2) & -z(y^2-a^2) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-z(y^2-a^2)) - \frac{\partial}{\partial z} y(x^2-z^2)\right] - j\left[\frac{\partial}{\partial x}(-z(y^2-a^2)) - \frac{\partial}{\partial z}(x(y^2-a^2))\right] + k\left[\frac{\partial}{\partial x} y(x^2-z^2) - \frac{\partial}{\partial y} x(y^2-a^2)\right]$$

$= i(-2zy + 2zy) - j(0-0) + k(2xy - 2xy)$

$= i(0) - j(0) + k(0) = 0$

Now $\bar{X} \cdot \text{curl}\,\bar{X} = x(y^2-a^2)(0) + y(x^2-z^2)(0) + (-z)(y^2-a^2)(0)$

$= 0$

∴ The given diff.. eq^n is integrable.

Divide with $(x^2-z^2)(y^2-a^2)$

$$\frac{x}{x^2-z^2}dx + \frac{y}{y^2-a^2}dy - \frac{z}{x^2-z^2}dz = 0$$

$P(x,z)dx + Q(y)dy + R(x,z)\,dz = 0$

$$\left(\frac{x\,dx - z\,dz}{x^2-z^2}\right) + \frac{y}{y^2-a^2}dy = 0$$

multiply and divide with 2

$$\frac{1}{2}\left(\frac{2x\,dx - 2z\,dz}{x^2-z^2}\right) + \frac{1}{2}\left(\frac{2y}{y^2-a^2}\right)dy = 0$$

$$\frac{1}{2}\frac{d(x^2-z^2)}{x^2-z^2} + \frac{1}{2}\frac{d(y^2-a^2)}{y^2-a^2} = 0$$

Now integrate on both sides

$\Rightarrow \frac{1}{2}\log(x^2-z^2) + \frac{1}{2}\log(y^2-a^2) = \log c$

$\Rightarrow \frac{1}{2}\log(x^2-z^2)(y^2-a^2) = \log c$

$\Rightarrow \log(x^2-z^2)(y^2-a^2) = 2\log c$

$\Rightarrow (x^2-z^2)(y^2-a^2) = 2c$

This is the req. solⁿ of given eqⁿ.

2. $2yz\,dx - 2xz\,dy - (x^2-y^2)(z-1)dz = 0$

Given:- $2yz\,dx - 2xz\,dy - (x^2-y^2)(z-1)dz = 0$ —(1) compare (1) with $P\,dx + Q\,dy + R\,dz = 0$

$X = (P, Q, R) = (2yz, -2xz, -(x^2-y^2)(z-1))$

$$\text{curl } \bar{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2yz & -2xz & -(x^2-y^2)(z-1) \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(-x^2z + x^2 + y^2z - y^2) - \frac{\partial}{\partial z}(-2xz)\right] - j\left[\frac{\partial}{\partial x}(-x^2z + x^2 + y^2z - y^2) - \frac{\partial}{\partial z}(2yz)\right] + k\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(2yz)\right]$

$= i(2yz - 2y + 2x) - j(-2xz + 2x \overset{-2y}{}) + k(-2z - 2z)$

Now $\bar{X}\cdot \text{curl }\bar{X} = 2yz(2yz - 2y + 2x) + (-2xz)(2xz - 2x + 2y) + (-4z)\cdot(-x^2z + x^2 + y^2z - y^2)$

$= \cancel{4y^2z^2} - \cancel{4y^2z} + \cancel{4xyz} - \cancel{4x^2z^2} + \cancel{4x^2z} - \cancel{4xyz} + \cancel{4x^2z^2} - \cancel{4zx^2} - \cancel{4y^2z^2} + \cancel{4y^2z}$

$= 0$

∴ The given diff. eqⁿ is integrable.

Divide with $z(x^2-y^2)$

$\frac{2y}{x^2-y^2}dx - \frac{2x}{x^2-y^2}dy - \frac{(z-1)}{z}dz = 0$

$P(x,y)dx + Q(x,y)dy + R(z)dz = 0$

Since by procedure, we have

$P\,dx + Q\,dy = 0$ is exact

ie, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Here $P = \frac{2y}{x^2-y^2}$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2-y^2}\right)$

$= \frac{(x^2-y^2)(2) - 2y(-2y)}{(x^2-y^2)^2}$

$= \frac{2x^2 - 2y^2 + 4y^2}{(x^2-y^2)^2}$

$\frac{\partial P}{\partial y} = \frac{2x^2+2y^2}{(x^2-y^2)^2}$

$Q = \frac{-2x}{x^2-y^2}$

$$\frac{dQ}{dx} = \frac{(x^2-y^2)(-2)-(-2x)(2x)}{(x^2-y^2)^2}$$

$$= \frac{-2x^2+2y^2+4x^2}{(x^2-y^2)^2} = \frac{2x^2+2y^2}{(x^2-y^2)^2}$$

$\therefore \frac{dQ}{dx} = \frac{dP}{dy}$ is exact

Substitude P & Q values in $Pdx+Qdy=0$

$$\frac{2y}{x^2-y^2}dx - \frac{2x}{x^2-y^2}dy = 0$$

$\Rightarrow 2ydx - 2xdy = 0$

$\Rightarrow ydx = xdy$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating on both sides

$x = yc$

Since $u(x,y,z)=c$

$\Rightarrow u(x,y)=c$

$\Rightarrow u(x,y) = x/y$

$\therefore$ The primitive of eq^n (1) is $u(x,y)+\int R(z)dz = c$

$$\frac{x}{y} - \int \frac{z-1}{z}dz = c$$

$$\frac{x}{y} - \int dz + \int \frac{1}{z}dz = c$$

$$\frac{x}{y} - z + \log z = c$$

This is the req. sol^n of given eq^n.

d) Homogenous method:- The eq^n $P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=0$ —(1) is said to be homogenous if the functions P, Q, & R are homogenous in 'x', y, z of the same degree 'n'

TO derive the sol^n of such an eq^n we make substitutions

$y = ux$ and $z = vx$ —(2)

Now differentiate with 'y' & 'z'

$$\left.\begin{aligned} dy &= udx+xdu \\ dz &= vdx+xdv \end{aligned}\right\} \text{—(3)}$$

Substitue 2,3 in (1), we get

$P(x,ux,vx)dx + Q(x,ux,vx)(udx+xdu) + R(x,ux,vx)(vdx+xdv)=0$

$P(x(1,u,v))dx + Q(x(1,u,v))(udx+xdu) + R(x(1,u,v))vdx+xdv=0$

Divide with 'x'

$\Rightarrow P(1,u,v)\frac{dx}{x} + Q(1,u,v)\left(\frac{udx+xdu}{x}\right) + R(1,u,v)\left(\frac{vdx+xdv}{x}\right)=0$ —(4)

For simplification, Let us take

$$A(u,v) = \frac{Q(1,u,v)}{P(1,u,v)+uQ(1,u,v)+Rv(1,u,v)}$$

$$B(u,v) = \frac{R(1,u,v)}{P(1,u,v)+uQ(1,u,v)+vR(1,u,v)}$$

Substitude these values in eq^n (4)

$$\frac{dx}{x} + A(u,v)\,du + B(u,v)\,dv = c$$

By integrating this, we get the req. sol^n

Q) Verify that the eq^n $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$ is integrable and find the sol^n.

Given eq^n:- $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$ —(1)

Compare (1) with $Pdx + Qdy + Rdz = 0$

where $(P, Q, R) = (yz(y+z),\ xz(x+z),\ xy(x+y))$

$\bar{x} = (yz(y+z)i + xz(x+z)j + k\,xy(x+y))$

$$\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z+yz^2 & x^2z+xz^2 & x^2y+xy^2 \end{vmatrix}$$

$= i\left(\frac{\partial}{\partial y}(x^2y+xy^2) - \frac{\partial}{\partial z}(x^2z+xz^2)\right) - j\left(\frac{\partial}{\partial x}(x^2y+xy^2) - \frac{\partial}{\partial z}(y^2z+yz^2)\right) + k\left(\frac{\partial}{\partial x}(x^2z+xz^2) - \frac{\partial}{\partial y}(y^2z+yz^2)\right)$

$= i(2xy+2xy-2xz-2xz) - j(2xy+2xy-2yz-2zy) + k(2xz+2zx-2yz-2yz)$

$\bar{x}\cdot\text{curl}\,\bar{x} = (y^2z+z^2y)(4xy-4xz) + (x^2z+xz^2)(-4xy+4yz) + (x^2y+xy^2)(4xz-4yz)$

$= \cancel{4xy^3z} + \cancel{4xy^2z^2} - \cancel{4xy^2z^2} - \cancel{4xyz^3} - \cancel{4x^3zy} - \cancel{4x^2yz^2} + \cancel{4x^2yz^2} + \cancel{4xz^3y} + \cancel{4x^3yz} + \cancel{4x^2y^2z} - \cancel{4x^2y^2z} - \cancel{4xy^3z}$

$= 0$

$\bar{x}\cdot\text{curl}\,\bar{x} = (y^2z+yz^2(0)) + x^2z+xz^2(0) + x^2y+xy^2(0)$

$= 0$

$\therefore$ The given eq^n is integrable.

Substitude $\left.\begin{array}{l} y = ux \\ z = vx \end{array}\right\}$ —(2)

Differentiate on both sides, we get

$\left.\begin{array}{l} dy = u\,dx + x\,du \\ dz = v\,dx + x\,dv \end{array}\right\}$ —(3)

substitude (2) and (3) in (1)

$ux(vx)(ux+vx)\,dx + x(vx)(x+vx)(u\,dx+x\,du) + x(ux)(x+ux)(v\,dx+x\,dv) = 0$

$\Rightarrow x^3[uv(u+v)dx + v(1+v)(u\,dx+x\,du) + u(1+u)(v\,dx+x\,dv)] = 0$

$\Rightarrow uv(u+v)dx + v(1+v)(u\,dx+x\,du) + u(1+u)(v\,dx+x\,dv) = 0$ —(4)

It's general form is $Pdx + Qdy + Rdz = 0$

where $P = uv(u+v)$; $Q = v(1+v)$; $R = u(1+u)$

$$\frac{dx}{x}+A(u,v)\,du+B(u,v)\,dv=0 \quad -(5)$$

where $A(u,v)=\dfrac{Q(1,u,v)}{P(1,u,v)+uQ(1,u,v)+vR(1,u,v)}$ and

$$B(u,v)=\frac{R(1,u,v)}{P(1,u,v)+uQ(1,u,v)+vR(1,u,v)}$$

compare (4) with $pdx+Qdy+Rdz=0$ then we get P,Q,R values.

$$A(u,v)=\frac{v(1+v)}{uv(u+v)+uv(1+v)+uv(1+u)}$$

$$B(u,v)=\frac{u(1+v)}{uv(u+v)+uv(1+v)+uv(1+u)}$$

Substitude A, B in ⑤

$$\frac{dx}{x}+\frac{v(1+v)}{uv(u+v)+uv(1+v)+uv(1+u)}du+\frac{u(1+u)}{uv(u+v)+uv(1+v)+uv(1+u)}dv=0$$

$$\frac{dx}{x}+\frac{\cancel{v}(1+v)}{u\cancel{v}(u+v+1+v)+1+u)}du+\frac{\cancel{u}(1+u)}{\cancel{u}v(u+v+1+v+1+u)}dv=0$$

$$\frac{dx}{\cancel{d}x}+\frac{(1+v)}{2u(u+v+1)}du+\frac{1+u}{2v(1+u+v)}dv=0$$

$$\frac{2dx}{x}+\frac{1+v}{u(u+v+1)}du+\frac{1+u}{v(1+u+v)}dv=0 \quad —(6)$$

To integrate the above eq^n we ~~get~~ use partial functions.

$$\frac{1+v}{u(u+v+1)}=\frac{A}{u}+\frac{B}{u+v+1}$$

$\Rightarrow 1+v=A(u+v+1)+Bu$

put $v=-1$

$\Rightarrow 0=Au+Bu$

$\Rightarrow u(A+B)=0$

$\Rightarrow A+B=0$

$\Rightarrow A=-B$

ie, $A=1$; $B=-1$

$$\left.\begin{aligned}\frac{1+v}{u(u+v+1)}&=\frac{1}{u}-\frac{1}{u+v+1}\\ \text{lly}\ \frac{1+u}{v(u+v+1)}&=\frac{1}{v}-\frac{1}{u+v+1}\end{aligned}\right\}—(7)$$

Substitude ⑦ in ⑥

$$\frac{2dx}{x}+\left(\frac{1}{u}-\frac{1}{u+v+1}\right)du+\left(\frac{1}{v}-\frac{1}{u+v+1}\right)dv=0$$

$$\Rightarrow 2\frac{dx}{x}+\frac{du}{u}-\frac{du}{u+v+1}+\frac{1}{v}dv-\frac{dv}{u+v+1}=0$$

$\Rightarrow 2\frac{dx}{x} + \frac{du}{u} + \frac{dv}{v} - \left(\frac{du+dv}{u+v+1}\right) = 0$

$\Rightarrow 2\frac{dx}{x} + \frac{du}{u} + \frac{dv}{v} - \frac{d(u+v+1)}{u+v+1} = 0$

Integrate on both sides

$2\log x + \log u + \log v - \log(u+v+1) = \log c$

$\cancel{\log}\left(\frac{x^2uv}{u+v+1}\right) = \cancel{\log} c$

$\frac{x^2uv}{u+v+1} = c$

we have $y = ux$; $z = vx$

$\frac{x^2\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)}{\frac{z}{x}+\frac{y}{x}+1} = c$

$\frac{yz}{\frac{x+y+z}{x}} = c$

$c = \frac{xyz}{x+y+z}$ where 'c' is constant

e, Natani's Method:- Let the given pfaffin differential eq^n is $P(x,$ $y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ —(1)

In this method, we treat the 'z' variable as constant and solve the resulting eq^n $Pdx + Qdy = 0$ —(2)

Suppose the sol^n of eq^n (2) is $\phi(x,y,z) = C_1$ where C_1 is a constant then eq^n ① becomes

$\phi(x,y,z) = \varphi(z)$ —(3)

where φ is a function of z alone.

To determine the function $\varphi(z)$ we have to give a fixed value for 'x'.

ie, Let $x = \alpha$ (say)

Then $\phi(\alpha,y,z) = \varphi(z)$ —(4) is the sol^n of given diff--eq^n

$Q(\alpha,y,z)dy + R(\alpha,y,z)dz = 0$ —(5)

Now we can find the sol^n for the eq^n ⑤

It is in the form $K(y,z) = C_2$ —(6)

Since eq^n ④ & ⑥ represents the general sol^n of the differential eq^n ⑤

∴ their must be equivalent

By using the methods of the terms of the first order partial differential eqn, we can solve.

$\therefore$ If we eliminate the variable 'y' b/w the eqn's ④ & ⑥ we obtained an expression for the function $\varphi(z)$

substituting this expression in eqn ④ we get the soln of the given pfaffin differential eqn.

1. (55) verify that the eqn $z(z+y^2)dx+z(z+x^2)dy-xy(x+y)dz=0$ is integrable and find its primitive by using Natani's method.

Given eqn:- $z(z+y^2)dx+z(z+x^2)dy-xy(x+y)dz=0$ —(1)

compare (1) with $Pdx+Qdy+Rdz=0$

where $P=z(z+y^2)$; $Q=z(x+x^2)$; $R=-xy(x+y)$

$\bar{x}=(z^2+y^2z,\ z^2+zx^2;\ -x^2y-xy^2)\ (i,j,k)$

$=(z^2+y^2z)i+(z^2+zx^2)j+(-x^2y-xy^2)k$

$$\text{curl}\,\bar{x}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2+y^2z & z^2+zx^2 & -x^2y-xy^2 \end{vmatrix}$$

$=i\left[\frac{\partial}{\partial y}(-x^2y-xy^2)-\frac{\partial}{\partial z}(z^2+zx^2)\right]-j\left[\frac{\partial}{\partial x}(-x^2y-xy^2)-\frac{\partial}{\partial z}(z^2+y^2z)\right]+k\left[\frac{\partial}{\partial x}(z^2+zx^2)-\frac{\partial}{\partial y}(z^2+y^2z)\right]$

$=i[(-x^2-2xy)-(2z+x^2)]-j(-2xy-y^2-2z-y^2)+k(2xz-2yz)$

$=i(-2x^2-2xy-2z)-j(-2xy-y^2-2z-y^2)+k(2xz-2yz)$

$\bar{x}\cdot\text{curl}\,\bar{x}=(z^2+y^2z)(-2x^2-2xy-2z)+(z^2+zx^2)(2xy+y^2+2z+y^2)+(-x^2y-xy^2)(2xz-2yz)$

$=-2x^2z^2-2x^2y^2z\ \cancel{-2xyz^2}\ \cancel{-2xy^3z}\ \cancel{-2z^3}\ \cancel{-2y^2z^2}\ \cancel{+2xyz^2}\ \cancel{+y^2z^2}\ \cancel{+2z^3}\ \cancel{+y^2z^2}+2x^3yz$
$\cancel{+2xyx^2z}\ \cancel{+2z^2x^2}\ \cancel{+x^2y^2z}\ \cancel{-2x^3yz}\ \cancel{-2x^2y^2z}\ \cancel{+2x^2yz}\ \cancel{+2xy^3z}$

$\therefore\ \bar{x}\cdot\text{curl}\,\bar{x}=\bar{0}$

$\therefore$ The given differential eqn is integrable.

An inspection of the eqn suggets that it is probably simplest to take $dy=0$

So taking 'y' as a constant $\Rightarrow dy=0$

Substitude $dy=0$ in eqn (1)

$z(x+y^2)dx-xy(x+y)dz=0$

$\Rightarrow z(z+y^2)dx=xy(x+y)dz$

$$\Rightarrow \frac{dx}{xy(x+y)} = \frac{dz}{z(z+y^2)}$$

$$\Rightarrow \frac{1}{y}\frac{dx}{x(x+y)} = \frac{dz}{z(z+y^2)} \quad —(2)$$

From eq^n (2), consider $\frac{1}{x(x+y)} = \frac{A}{x} + \frac{B}{x+y}$

$$1 = A(x+y) + Bx$$

Let $B = -A$

$$1 = Ax + Ay - Ax$$

$$Ay = 1 \Rightarrow \boxed{A = \frac{1}{y}}$$

Since $B = -A$

$$\Rightarrow B = -1/y$$

$$\therefore \frac{1}{x(x+y)} = \frac{\frac{1}{y}}{x} + \frac{-\frac{1}{y}}{x+y} \quad —(3)$$

From (2) consider, $\frac{1}{z(z+y^2)} = \frac{A}{z} + \frac{B}{z+y^2}$

$$\Rightarrow 1 = A(z+y^2) + Bz$$

$$\Rightarrow 1 = Az + Ay^2 + Bz$$

Substitude $B = -A$

$$\Rightarrow 1 = Az + Ay^2 - Az$$

$$\Rightarrow A = \frac{1}{y^2}$$

$\because B = -A \Rightarrow B = -1/y^2$

$$\therefore \frac{1}{z(z+y^2)} = \frac{\frac{1}{y^2}}{z} \quad \frac{-\frac{1}{y^2}}{z+y^2} \quad —(4)$$

Substitude (3) and (4) in (2)

$$\frac{1}{\cancel{y}}\frac{1}{y}\left(\frac{1}{x} - \frac{1}{x+y}\right)dx = \frac{1}{\cancel{z}}\left(\frac{1}{y^2}\left(\frac{1}{z} - \frac{1}{z+y^2}\right)dz\right.$$

$$\left(\frac{1}{x} - \frac{1}{x+y}\right)dx = \frac{1}{\cancel{z}}\left(\frac{1}{z} - \frac{1}{z+y^2}\right)dz$$

$$\frac{dx}{x} - \frac{dx}{x+y} = \frac{dz}{z} - \frac{d(z+y^2)}{z+y^2}$$

Integrate on both sides

$$\log x - \log(x+y) = \log z - \log(z+y^2)$$

$$\log\left(\frac{x}{x+y}\right) - \log\left(\frac{z}{z+y^2}\right) = \log c$$

$$\left(\frac{x}{x+y}\right)\left(\frac{z+y^2}{z}\right) = c = f(y) \quad —(5)$$

It is req. sol^n of given diff. eq^n.

Another solution:- Let 'z' is a constant

$\Rightarrow dz = 0$

Substitude $dz=0$ in ①

$$z(z+y^2)\,dx + z(z+x^2)\,dy = 0$$

$$(z+y^2)dx + (z+x^2)\,dy = 0$$

Substitude $z=1$

$$(1+y^2)\,dx + (1+x^2)\,dy = 0$$

$$(1+y^2)\,dx = -(1+x^2)\,dy$$

$$\frac{dx}{1+x^2} = \frac{-dy}{1+y^2}$$

Integrate on both sides

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}c \qquad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right)$$

$$c = \frac{x+y}{1-xy} \Rightarrow \frac{1-xy}{x+y} = \frac{1}{c}$$

$$\Rightarrow \frac{1-xy}{x+y} = c_1 \quad -(6)$$

Substitude $z=1$ in ⑤

$$\frac{x(1+y^2)}{x+y} = f(y)$$

consider $f(y) + y c_2 = \dfrac{x(1+y^2)}{x+y} + y\left(\dfrac{1-xy}{x+y}\right)$

$$= \frac{x + \cancel{xy^2} + y - \cancel{xy^2}}{x+y}$$

$\therefore f(y) + y c_2 = 1$

$\Rightarrow f(y) = 1 - y c_2$

Substitude $f(y) = 1 - y c_2$ in ⑤

$$\left(\frac{x}{x+y}\right)\left(\frac{z+y^2}{z}\right) = 1 - y c_2$$

$$\therefore x(z+y^2) = z(x+y)(1-yc_2)$$

It is the req. solⁿ of given diff. eqⁿ.

Method-F:- Reduction to an ordinary Differential Equation:-

In this method, we reduce the problem of finding the solⁿ of a pfaffin differential equation of the type.

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0 \quad -(1)$$

for that of integrating one ordinary differential eqⁿ of the first order in two variables.

It is necessary that the condition for integrability should be satisfied.

If eq^n ① is integrable, it has a sol^n of the form $f(x,y,z)$ $= c$ —(2) representing one parameter family of surfaces in sp

These integral surfaces will be intersected in a single infinity of curves by the plane $z = x + ky$ —(3) where 'k' is constant.

The curves so form the sol^{n}s of the differential equation $P(x,y,k)\,dx + Q(x,y,k)\,dy = 0$ —(4) formed by eliminating z b/w the eq^n's (1) and (3)

Suppose the general sol^n of eq^n (4) is $\phi(x,y,k) =$ constant

Since a point on the axis of the plane $f(x,y,z) = c$ is determined by $y = 0, x = c$ (a constant)

Then we must have $\phi(x,y,k) = \phi(c,0,k)$ —(5)

If we eliminate 'k' b/w eq^n's ⑤ & ② we obtain the integral surfaces required in the form $\phi(x,y,\frac{z-x}{y}) = \phi(c,0,\frac{z-x}{y})$

1, Integrating the eq^n $(y+z)dx + (z+x)dy + (x+y)dz = 0$ and reduce it to an ordinary differential equation.

Given eq^n :- $(y+z)dx + (z+x)dy + (x+y)dz = 0$ —(1)

Compare (1) with $P dx + Q dy + R dz = 0$

where $P = y+z$; $Q = z+x$; $R = x+y$

$\bar{x} = (y+z, z+x, x+y)\,(i,j,k)$

$= (y+z)\,i + (z+x)\,j + (x+y)\,k$

$$\text{curl}\,\bar{x} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ y+z & z+x & x+y \end{vmatrix}$$

$$= i\left[\frac{d}{dy}(x+y) - \frac{d}{dz}(z+x)\right] - j\left[\frac{d}{dx}(x+y) - \frac{d}{dz}(y+z)\right] + k\left[\frac{d}{dx}(z+x) - \frac{d}{dy}(y+z)\right]$$

$$= i(1-1) - j(1-1) + k(1-1) = i(0) - j(0) + k(0) = 0$$

$\therefore \text{curl}\,\bar{x} = 0$

$\therefore \bar{x}\cdot\text{curl}\,\bar{x} = 0$

$\therefore$ The given diff.. eq^n is integrable.

Substitude $z = x + ky$

$dz = dx + k\,dy$

Substitude these values in ①

$(y+x+ky)\,dx+(x+ky+x)\,dy+(x+y)(dx+k\,dy)=0$

$y\,dx+x\,dx+ky\,dx+x\,dy+ky\,dy+x\,dy+x\,dx+xk\,dy+y\,dx+yk\,dy=0$

$\cancel{x dy}+\cancel{y dx}+\cancel{x dx}+\cancel{2k y dy}+\cancel{k}(\cancel{y dx}+\cancel{x dy}+\cancel{x dx}+\cancel{y}$

$(y+x+ky+x+y)\,dx+dy\,(x+ky+x+xk+yk)=0$

$(2x+2y+ky)\,dx=-dy\,(2x+kx+2ky)=0$

$$\frac{dy}{dx}=\frac{-(2x+2y+ky)}{2x+kx+2ky} \quad -(2)$$

Put $y=ux$

$$\frac{dy}{dx}=u+x\frac{du}{dx} \quad -(3)$$

From (2) and (3) $u+x\dfrac{du}{dx}=\dfrac{-(2x+2y+ky)}{2x+kx+2ky}$

$$u+x\frac{du}{dx}=\frac{-(2x+2ux+kux)}{2x+kx+2k(ux)}$$

$$=\frac{-\cancel{x}(2+2u+ku)}{x(2+k+2ku)}$$

$$u+\frac{2+2u+ku}{2+k+2ku}=-x\frac{du}{dx}$$

$$\frac{2u+2ku+2ku^2+2+2u+ku}{2+k+2ku}=-x\frac{du}{dx}$$

$$\frac{2+4u+2ku+2ku^2}{2+k+2ku}=-x\frac{du}{dx} \Rightarrow \frac{2(1+2u+ku+ku^2)}{2+k+2ku}=-x\frac{du}{dx}$$

Separate the variables

$$\frac{2ku+k+2}{ku^2+ku+2u+1}\,du=-2\frac{dx}{x}$$

$$\frac{d(ku^2+ku+2u+1)}{ku^2+ku+2u+1}=-2\frac{dx}{x}$$

Integrate on both sides

$\log(ku^2+ku+2u+1)=-2\log x+\log c$

$x^2(ku^2+ku+2u+1)=c$

Since $\phi(x,y,z)=c$

$\Rightarrow \phi(x,y,k)=x^2(ku^2+ku+2u+1)$

since $y=ux$

$\Rightarrow u=y/x$

Substitude $u=y/x$ in above eq^n.

$$\phi(x,y,k)=x^2\left(k\frac{y^2}{x^2}+k\frac{y}{x}+2\frac{y}{x}+1\right)=\cancel{x^2}\left(\frac{ky^2+kxy+2xy+x^2}{\cancel{x^2}}\right)$$

$=ky^2+kxy+2xy+x^2$

$\phi(x, y, k) = ky(y+x) + 2xy + x^2$ ——(4)

Since $z = x + ky$

$ky = z - x$

$\Rightarrow k = \frac{z-x}{y}$

Now eliminating 'k' from eqⁿ (4)

(4) $\Rightarrow \phi\left(x, y, \frac{z-x}{y}\right) = (z-x)(y+x) + 2xy + x^2$

$= yz + xz - xy - \cancel{x^2} + 2xy + \cancel{x^2}$

$= xz + zy + xy$

$\therefore \phi\left(x, y, \frac{z-x}{y}\right) = xz + xy + zy$ is the required solⁿ of given eqⁿ.

chapter-2

Section-I

PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Introduction:- We know procedure to the study of partial differ-ential equations arise in geometry and physics when the number of independent variables in the problem under the dis-cussion is two or more.

When such is the case, any dependent variable is likely to be a function of more than one variable, so that it poss-es not ordinary derivatives with respect to a single variable but partial derivatives with respect to several variables.

For instance, in the study of the thermal effect in a solid body the temperature 'θ' may vary from point to point in the solid as well as from time to time and as a conseq-uence, the derivatives $\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}, \frac{\partial \theta}{\partial t}$

Partial Differential Eqⁿ:- An eqⁿ containing one or more partial derivatives of the unknown functions of two or more variables is known as partial diff---al eqⁿ.

Ex:- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

Order of a Partial differential Eqⁿ:- The order of a partial differential eqⁿ is the order of highest derivative occuring in the eqⁿ.

Ex:- 1, $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$ is of 1^{st} order.

2, $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial x}\right)^{3/2}$ is of second order.

3, $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = \partial x - \frac{\partial z}{\partial x}$ is of third order.

Degree of a partial differential Eq^n:- The degree of a partial differential eq^n is the degree of the highest derivative occurring in the eq^n after the equation has been expressed ie, made free from radicals and fractions as far as the derivatives are concerned.

Ex:- 1, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$ is of first degree.

2, $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = \partial x - \frac{\partial z}{\partial x}$ is of first degree.

For example, if we take x, y as dependent variables and 't' to be independent variable then the eq^n.

$\frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t}$ —① is a second order eq^n in two variables.

The eq^n $\left(\frac{\partial \theta}{\partial x}\right)^3 + \frac{\partial \theta}{\partial t} = 0$ is a first order eq^n in two variables.

The eq^n $x\frac{\partial \theta}{\partial x} + y\frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial t} = 0$ is a 1^{st} order eq^n in three variables.

First Order partial differential Equation:- Suppose there are two independent variables 'x' & 'y' and one dependent variable 'z' if we write $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$, then the 1^{st} order partial differential eq^n can be written in the symbolic form $f(x, y, z, p, q) = 0$

ORIGIENCE OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

we know that the first order partial differential eq^n is

$f(x, y, z, p, q) = 0$ —(1)

Hence $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

Sol^n's of first order Partial differential Eq^n:- Before discussing the sol^{n}s of equations of type ① we shall examine the interesting question of how their arise. Suppose that we consider the eq^n's

$x^2+y^2+(z-c)^2=a^2$ — (2) where 'a' & 'c' are arbitrary consta

The eq^n (2) represents the set of spheres whose centre lie along the z-axis.

Differentiate with respect to 'x'

$$2x + 2(z-c)\frac{dz}{dx} = 0$$

we know that $\frac{dz}{dx} = p$

$\Rightarrow 2x + 2p(z-c) = 0$

$\Rightarrow x + pz - pc = 0$ — (3)

Differentiate (2) with respect to 'y'

$$2y + 2(z-c)\frac{dz}{dy} = 0$$

we know that $\frac{dz}{dz} = q$

$\Rightarrow 2y + 2(z-c)q = 0$

$\Rightarrow y + (z-c)q = 0$

$\Rightarrow y + zq - cq = 0$ — (4)

Now eliminating the arbitrary constant 'c' from (3) & (4)

(3) $\Rightarrow x + pz = pc$

$$c = \frac{x+pz}{p} \quad -(5)$$

(4) $\Rightarrow y + zq = cq$

$$c = \frac{y+zq}{q} \quad -(6)$$

from (5) & (6) we have $\frac{x+pz}{p} = \frac{y+zq}{q}$

$$q(x+pz) = p(y+zq)$$

$$qx + \cancel{pq}z = py + \cancel{pq}z$$

$py - qx = 0$ which is of 1^{st} order. $\rightarrow$ (A)

In some sense that the set of all spheres with centres on the z-axis characterised by the partial differential eq^n (A)

(1) Set of all right circular cones:-

If we take the set of all spheres whose centres lie on the z-axis and the geometrical entries, the solutions can be described by the same equation.

For example, the equation $x^2+y^2=(z-c)^2\tan^2\alpha$ —(1) where 'c' and 'α' arbitrary constants.

Eq^n (1) represents the set of all right circular cones whose axes coincide with line OZ.

Now differentiating (1) with respect to 'x'

$$2x \mp 2(z-c)\frac{dz}{dx}\tan^2\alpha$$

we know that $\frac{dz}{dx} = p$

$$2x = 2(z-c)p\tan^2\alpha$$

$$2x = 2zp\tan^2\alpha - 2pc\tan^2\alpha$$

$$x = zp\tan^2\alpha - pc\tan^2\alpha$$

$$c\tan^2\alpha = \frac{zp\tan^2\alpha - x}{p} \quad \text{—(2)}$$

Now differentiating (1) with respect to 'y'

$$\cancel{2}y = \cancel{2}(z-c)\frac{dz}{dy}\tan^2\alpha$$

$$y = (z-c)q\tan^2\alpha \quad (\because \frac{dz}{dy} = q)$$

$$y = zq\tan^2\alpha - cq\tan^2\alpha$$

$$c\tan^2\alpha = \frac{zq\tan^2\alpha - y}{q} \quad \text{—(3)}$$

Now eliminating the constants 'c' and 'α' from eq^n's (2) & (3) we have, $\frac{zp\tan^2\alpha - x}{p} = \frac{zq\tan^2\alpha - y}{q}$

$$pzq\cancel{\tan^2\alpha} - xq = pq\cancel{z}\tan^2\alpha - py$$

$$\boxed{py - qx = 0} \quad \text{—(A)}$$

which is of first order.

In some sense, the set of all spheres with centres on the z-axis is characterized by the P.D.E —(A)

(4) Spheres :- Now we have to find the eq^n's of surfaces of revolution in which spheres and cones have in common and having the surface of revolution along Oz axis.

So, these eq^n's are characterized by the form $z = f(x^2+y^2)$ —(1)

where 'f' is arbitrary.

Let $x^2+y^2 = u$

Differentiate (1) with respect to 'x'

$$\frac{dz}{dx} = f'(x^2+y^2)\cdot 2x$$

we know that $\frac{dz}{dx} = p$ and $x^2+y^2 = u$

$$p = f'(u)\cdot 2x$$

$$f'(u) = \frac{p}{2x} \quad \text{--- (2)}$$

Differentiate (1) with respect to 'y'

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\, 2y$$

we know that $\frac{\partial z}{\partial y} = q$ and $x^2+y^2 = u$

$\Rightarrow q = f'(u)\, 2y$

$\Rightarrow f'(u) = \frac{q}{2y}$ --- (3)

Since 'f' is arbitrary.

Now eliminate f' from eqn's (2) & (3)

$$\Rightarrow \frac{p}{\cancel{2}x} = \frac{q}{\cancel{2}y}$$

$\Rightarrow py = qx$

$\Rightarrow \boxed{py - qx = 0} \longrightarrow (A)$

In this case also eqn (A) is satisfied.

Thus, we see that the function 'z' defined by each of the eq-
-uations $x^2+y^2+(z-c)^2 = a^2$

$$x^2+y^2 = (z-c)^2 \tan^2\alpha$$

$z = f(x^2+y^2)$ are in same sense that these three equations have common soln.

ie, $py - qx = 0$

<u>Note:-</u> We shall now generalize this argument straightly for relations $x^2+y^2+(z-c)^2 = a^2$ & $x^2+y^2 = (z-c)^2 \tan^2\alpha$ are both of the type $F(x, y, z, a, b) = 0$ --- (1) where a & b denote arbitrary constants.

Now differentiate (1) with respect to 'x'

$$\frac{\partial F}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial F}{\partial z} = 0$$

we know that $\frac{\partial z}{\partial x} = p$

$$\Rightarrow \frac{\partial F}{\partial x} + p \cdot \frac{\partial F}{\partial z} = 0 \quad \text{--- (2)}$$

Now differentiate (1) with respect to 'y'

$$\frac{\partial F}{\partial y} + \frac{\partial z}{\partial y} \cdot \frac{\partial F}{\partial z} = 0$$

we know that $\frac{\partial z}{\partial y} = q$.

$$\frac{\partial F}{\partial y} + q\, \frac{\partial F}{\partial z} = 0 \quad \text{--- (3)}$$

ie, $\left.\begin{aligned} \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \\ \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \end{aligned}\right\} \quad -(4)$

The set of eqn's (1) & (4) contribute three eqn's involving two arbitrary constants 'a' and 'b'.

In general case, it will be possible to eliminate 'a' and 'b' from these eqn's to obtain a relation of the kind $f(x,y,z,p,q)=0$ showing that the system of surfaces $x^2+y^2+(z-c)^2 = a^2$ gives rise to a partial differential equation of first order.

Note:- The obvious generalization of the relation $z=f(x^2+y^2)$ is a relation between x, y, z of the type $F(u,v)=0$ where 'u' and 'v' are known functions of 'x' and 'y' and z, F are arbitrary functions of u and 'v'.

Solution of the surface $F(u,v)=0$:-

consider the surface equation $F(u,v)=0$ — (1) where 'u' and 'v' are known functions of x, y, z and F is an arbitrary function of 'u' and 'v' ie, $F(u(x,y,z), v(x,y,z))=0$

Now differentiate (1) with respect to 'x'.

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}[u_x + p u_z] + \frac{\partial F}{\partial v}[v_x + p v_z] = 0 \quad \left(\because \frac{\partial z}{\partial x} = p\right) \quad —(2)$$

Now differentiate (1) with respect to 'y'.

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial z}\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\cdot\frac{\partial v}{\partial z}\right] = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}[u_y + q u_z] + \frac{\partial F}{\partial v}[v_y + q v_z] = 0 \quad \left(\because \frac{\partial z}{\partial y} = q\right) \quad —(3)$$

Now, eliminate the arbitrary functions $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$

From (2) and (3)

To get this eliminated eqn, we multiply eqn (2) with $(u_y + q u_z)$ and (3) with $(u_x + p u_z)$

Now, $(2) \times (u_y + q u_z) - (3) \times (u_x + p u_z) = 0$

$$\frac{\partial F}{\partial u}(u_x + p u_z)\cancel{(u_y + q u_z)} + \frac{\partial F}{\partial v}(v_x + p v_z)(u_y + q u_z) - \frac{\partial F}{\partial u}\cancel{(u_y + q u_z)}(u_x + p u_z)$$
$$- \frac{\partial F}{\partial v}(v_y + q v_z)(u_x + p u_z) = 0$$

$\Rightarrow \frac{\partial F}{\partial v}(v_x + p v_z)(u_y + q u_z) - \frac{\partial F}{\partial v}(v_y + q v_z)(u_x + p u_z) = 0$

$\Rightarrow \frac{\partial F}{\partial v}\left[u_y v_x + q u_z v_x + p v_z u_y + p\cancel{q v_z u_z} - v_y u_x - v_y p u_z - q v_z u_x - \cancel{pq v_z u_z}\right] = 0$

$\Rightarrow \frac{\partial F}{\partial v}\left[(u_y v_x + (-v_y u_x)) + p(v_z u_y - v_y u_z) + q(u_z v_x - v_z u_x)\right] = 0$

$\Rightarrow \frac{\partial F}{\partial v} \cdot p(v_z u_y - v_y u_z) + \frac{\partial F}{\partial v} q(u_z v_x - v_z u_x) = -\frac{\partial F}{\partial v}(u_y v_x - v_y u_x)$

$\Rightarrow \frac{\partial F}{\partial v} p(v_z u_y - v_y u_z) + \frac{\partial F}{\partial v} q(u_z v_x - v_z u_x) = \frac{\partial F}{\partial v}(v_y u_x - u_y v_x)$

$$\Rightarrow p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

This is a first order partial differential equation satisfied by the surface $F(u,v)=0$ where $\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - v_x u_y$

Ex:- Consider the set of all spheres unit radius with centre in the xy plane is $(x-a)^2+(y-b)^2+z^2=1$ then find the partial differential equation.

Sol:- Given eqn is $(x-a)^2+(y-b)^2+z^2=1$ — (1)

To find the required partial differential equation, we eliminate the arbitrary constants 'a' and 'b'

Now differentiate (1) with respect to 'x'

$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$

$\Rightarrow (x-a) + zp = 0 \quad (\because \frac{\partial z}{\partial x} = p)$

$\Rightarrow x - a = -pz$ — (2)

Differentiate (1) with respect to 'y'

$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$

$\Rightarrow (y-b) + zq = 0 \quad (\because \frac{\partial z}{\partial y} = q)$

$\Rightarrow y - b = -qz$ — (3)

Sub. (2) and (3) in (1)

$(1) \Rightarrow (-pz)^2 + (-qz)^2 + z^2 = 1$

$\Rightarrow p^2z^2 + q^2z^2 + z^2 = 1$

$\Rightarrow z^2(p^2+q^2+1) = 1$ is the required solution.

Problems:-

1, Eliminate the constants 'a' and 'b' from the following eqⁿ's and the find the corresponding partial differential equation $z=(x+a)(y+b)$

Sol: Given equation is $z=(x+a)(y+b)$ —(1)

Differentiate (1) with respect to 'x'

$$\frac{\partial z}{\partial x} = 1(y+b) \quad (\because \frac{\partial z}{\partial x}=p)$$

$$p = y+b \text{ —(2)}$$

Differentiate (1) with respect to 'y'

$$\frac{\partial z}{\partial y} = (x+a)1 \quad (\because \frac{\partial z}{\partial y}=q)$$

$$q = x+a \text{ —(3)}$$

Substitude (2) & (3) in (1)

(1) ⇒ $z=pq$ is the req. P.D.E.

2, $2z=(ax+y^2)^2+b$

Given eqⁿ :- $2z=(ax+y^2)^2+b$ →(1) where a, b are arbitrary constants

To find the req. P.D.E. we have to eliminate the arbitrary constants 'a' and 'b'

Differentiate (1) with respect to 'x'

$$\cancel{2}\left(\frac{\partial z}{\partial x}\right)=\cancel{2}(ax+y)a$$

$$p = a(ax+y) \text{ —(2)} \quad (\because \frac{\partial z}{\partial x}=p)$$

Differentiate (1) with respect to 'y'

$$\cancel{2}\frac{\partial z}{\partial y}=\cancel{2}(ax+y)$$

$$q = ax+y \text{ —(3)} \quad (\because \frac{\partial z}{\partial y}=q)$$

Now multiply (2)×x & (3)×y and adding these two equations, we get

$$px+qy = ax(ax+y)+y(ax+y)$$

$$=(ax+y)(ax+y)$$

$$=q\,q \text{ (by (3))}$$

$px+qy=q^2$ is the req. P.D.E.

3, $ax^2+by^2+z^2=1$

Given eqⁿ :- $ax^2+by^2+z^2=1$ —(1)

where a, b are arbitrary constants.

To find the req. P.D.E. we have, to eliminate the arbitrary constants 'a' and 'b'

Differentiate (1) with respect to 'x'

$$2ax+2z\frac{\partial z}{\partial x}=1$$

$ax + pz = 0$ — (2) $\left(\because \frac{\partial z}{\partial x} = p\right)$

Differentiate (1) with respect to 'y'

$2by + 2z\frac{\partial z}{\partial y} = 1$ $\left(\because \frac{\partial z}{\partial y} = q\right)$

$by + zq = 1$ — (3)

From (2), $ax = -pz \Rightarrow a = \frac{-pz}{x}$

From (3), $by = -zq \Rightarrow b = \frac{-zq}{y}$

Substitude 'a' & 'b' in (1)

$\frac{-pz}{\cancel{x}}(x^{\cancel{2}}) + \left(\frac{-zq}{\cancel{y}}\right)(y^{\cancel{2}}) + z^2 = 1$

$-pzx - zqy + z^2 = 1$

$\Rightarrow -z[px + qy - z] = 1 \Rightarrow px + qy - z = \frac{1}{-z}$

4, $z = x + ax^2y^2 + b$

Given eq^n:- $z = x + ax^2y^2 + b$ — (1)

Differentiate (1) with respect to 'x'

$\frac{\partial z}{\partial x} = 2axy^2 + 1$ $\left(\because \frac{\partial z}{\partial x} = p\right)$

$p = 1 + 2axy^2$ — (2)

Differentiate (1) with respect to 'y'

$\frac{\partial z}{\partial y} = 2ax^2y$ $\left(\because \frac{\partial z}{\partial y} = q\right)$

$q = 2ax^2y$ — (3)

From (2) and (3) $2axy^2 = p-1 \Rightarrow a = \frac{p-1}{2xy^2}$ — (4)

$2ax^2y = q \Rightarrow a = \frac{q}{2x^2y}$ — (5)

From (4) & (5) $\frac{p-1}{\cancel{2}\cancel{x}y^{\cancel{2}}} = \frac{q}{\cancel{2}x^{\cancel{2}}\cancel{y}}$

$\frac{p-1}{y} = \frac{q}{x} \Rightarrow px - x + (qy) = 0$

$\Rightarrow px - x - qy = 0$

5, $z = ax + by$

Given eq:- $z = ax + by$ — (1) where a, b are arbitrary constants.

To find the req. P.D.Eq^n we have to eliminate

Differentiate (1) with respect to 'x'

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$

Differentiate (1) with respect to 'y'

$\frac{\partial z}{\partial y} = b \Rightarrow q = b$

Substitude a, b in eq^n (1)

$z = px + qy$ is the req. sol^n of P.D.E.

6) $z^2(1+a^3) = 8(x+ay+b)^3$

Given eqn:- $z^2(1+a^3) = 8(x+ay+b)^3$ — (1)

Differentiate (1) with respect to 'x'

$$\cancel{2}z\frac{\partial z}{\partial x}(1+a^3) = \overset{12}{\cancel{24}}(x+ay+b)^2$$

$$zp(1+a^3) = 12(x+ay+b)^2 \quad \left(\because \frac{\partial z}{\partial x} = p\right) \text{ — (2)}$$

Differentiate (1) with respect to 'y'

$$\cancel{2}z\frac{\partial z}{\partial y}(1+a^3) = \overset{12}{\cancel{24}}a(x+ay+b)^2$$

$$zq(1+a^3) = 12a(x+ay+b)^2 \quad \left(\because \frac{\partial z}{\partial y} = q\right) \text{ — (3)}$$

Now cubing and adding (2) and (3)

$$z^3p^3(1+a^3)^3 + z^3q^3(1+a^3)^3 = 12^3(x+ay+b)^6 + (12)^3a^3(x+ay+b)^6$$

$$z^3(1+a^3)^{\cancel{3}\,2}(p^3+q^3) = 12^3(x+ay+b)^6\cancel{(1+a^3)}$$

$$z^3(1+a^3)^2(p^3+q^3) = (12)^3(x+ay+b)^6$$

multiply 'z' on both sides

$$z^4(1+a^3)^2(p^3+q^3) = (12)^3z(x+ay+b)^6$$

$$\left(z^2(1+a^3)\right)^2(p^3+q^3) = (12)^3z(x+ay+b)^6$$

From 1 we have $z^2(1+a^3) = 8(x+ay+b)^3$

$$(p^3+q^3)\left(8(x+ay+b)^3\right)^2 = 1728z(x+ay+b)^6$$

$$(p^3+q^3)\,64\cancel{(x+ay+b)^6} = 1728z\cancel{(x+ay+b)^6}$$

$$64(p^3+q^3) = 1728z$$

$\Rightarrow p^3+q^3 = 27z$ is the req. P.D.E.

Eliminate The Arbitrary Function 'f' from the following Equations

1) $z = xy + f(x^2+y^2)$

Given equation is $z = xy + f(x^2+y^2)$ — (1)

Let $v = x^2+y^2$

$\Rightarrow z = xy + f(v)$ — (2)

Partially differentiate (2) with respect to 'x'

$$\frac{\partial z}{\partial x} = y + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

we know that $\frac{\partial z}{\partial x} = p$

$$\Rightarrow p = y + f_v\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

$$\Rightarrow p = y + f_v(2x)$$

$$f_v = \frac{p-y}{2x} \quad —(3)$$

Partially differentiate (2) with respect to 'y'

$$\frac{\partial z}{\partial y} = x + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

we know that $\frac{\partial z}{\partial y} = q$

$$\Rightarrow q = x + f_v \frac{\partial}{\partial y}(x^2+y^2)$$

$$\Rightarrow q = x + f_v (2y)$$

$$f_v = \frac{q-x}{2y} \quad —(4)$$

Now eliminate the arbitrary function 'f' from (3) and (4)

$$\frac{p-y}{\cancel{2}x} = \frac{q-x}{\cancel{2}y}$$

$$py - y^2 = qx - x^2$$

$\Rightarrow x^2 - y^2 - qx + py = 0$ is the required P.D.E.

3, $z = x + y + f(xy)$

Given eqn :- $z = x + y + f(xy)$ —(1)

Let $v = xy$

$$z = x + y + f(v) \quad —(2)$$

Differentiate (2) with respect to 'x' partially.

$$\frac{\partial z}{\partial x} = 1 + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \qquad (\because \frac{\partial z}{\partial x} = p)$$

$$p = 1 + f_v \frac{\partial}{\partial x}(xy)$$

$$p = 1 + f_v (y)$$

$$f_v = \frac{p-1}{y} \quad —(3)$$

Differentiate (2) with respect to 'y' partially

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + 1$$

$$q = 1 + f_v \frac{\partial}{\partial y}(xy)$$

$$q = 1 + f_v x$$

$$f_v = \frac{q-1}{x} \quad —(4)$$

Now eliminate the arbitrary function 'f' from (3) and (4)

$$\frac{p-1}{y} = \frac{q-1}{x}$$

$$px - x = qy - y$$

$$x - y + qy - px = 0$$

$\therefore x - y - px + qy$ is the required P.D.E.

3. $z = f\left(\frac{xy}{z}\right)$

Given eq^n is $z = f\left(\frac{xy}{z}\right)$ — (1)

Let $v = \frac{xy}{z}$

$\Rightarrow z = f(v)$ — (2)

Differentiate (2) with respect to 'x' partially

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$p = f_v \frac{\partial v}{\partial x}$$

$$p = f_v v_x \quad \text{— (3)}$$

consider $v_x = \frac{\partial}{\partial x}(v)$

$$= \frac{\partial}{\partial x}\left(\frac{xy}{z}\right)$$

$$= \frac{y \cdot z - xy\frac{\partial z}{\partial x}}{z^2}$$

$$= \frac{yz - xyp}{z^2} \quad \left(\because \frac{\partial z}{\partial x} = p\right) \text{— (4)}$$

Substitude (4) in (3)

$$p = f_v \frac{yz - xyp}{z^2}$$

$$\Rightarrow f_v = \frac{pz^2}{yz - xyp} \quad \text{— (5)}$$

Differentiate (2) with respect to 'y' partially

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$q = f_v \cdot v_y \quad \text{— (6)}$$

Now, $v_y = \frac{\partial}{\partial y}(v) = \frac{\partial}{\partial y}\left(\frac{xy}{z}\right)$

$$= \frac{xz - xy\frac{\partial z}{\partial y}}{z^2}$$

$$= \frac{xz - xyq}{z^2} \quad \text{— (7)} \left(\because \frac{\partial z}{\partial y} = q\right)$$

Substitude (7) in (6)

$$q = f_v \cdot \left(\frac{xz - xyq}{z^2}\right)$$

$$f_v = \frac{qz^2}{xz - xyq} \quad \text{— (8)}$$

Eliminate the arbitrary function 'f' from (5) and (8)

$$\frac{p\cancel{z^2}}{yz - xyp} = \frac{q\cancel{z^2}}{xz - xyq}$$

$$pxz - xy\cancel{pq} = qyz - xy\cancel{pq}$$

$\therefore z(xp-yq)$ is the required P.D.E.

4. $z = f(x-y)$

Given equation:- $z = f(x-y)$ —(1)

Let $v = x-y$

$z = f(v)$ —(2)

Differentiate (2) with respect to 'x' partially.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$p = f_v \frac{\partial}{\partial x}(x-y)$$

$$p = f_v \quad —(3)$$

Differentiate (2) with respect to 'y' partially

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$q = f_v \frac{\partial}{\partial y}(x-y) \quad \left(\because \frac{\partial z}{\partial y} = q\right)$$

$$q = f_v(-1)$$

$$q = -f_v \quad —(4)$$

Eliminate the arbitrary function 'f' from (3) and (4)

$\Rightarrow p = -q$

$\Rightarrow p+q = 0$ is the required P.D.E.

5. $f(z-xy, x^2+y^2) = 0$

Given eq^n is $f(z-xy, x^2+y^2) = 0$ —(1)

Let $u = z-xy$ and $v = x^2+y^2$ —(2)

$\Rightarrow f(u, v) = 0$ be the surface eq^n —(3)

By known theorem, if $f(u,v)=0$ be the surface eq^n then P.D.E. for this surface eq^n is

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$p[u_y v_z - u_z v_y] + q[u_z v_x - v_z u_x] = u_x v_y - u_y v_x \quad —(4)$$

Since $u = z-xy$ $\quad v = x^2+y^2$

$u_x = -y$ $\quad v_x = 2x$

$u_y = -x$ $\quad v_y = 2y$

$u_z = 1$ $\quad v_z = 0$

Substitute all these values in eq^n (4), we get

$\Rightarrow p[-x(0)-1(2y)]+q[1(x^2 2y)-0(-y)] = -y(2y)+x(2x)$

$\Rightarrow -2yp+2xq = 2x^2-2y^2$

$\Rightarrow x^2-y^2+yp-xq=0$ is the required P.D.E.

6 $F\left(\frac{xy}{z}, \frac{x-y}{z}\right)=0$ where 'z' is not function of 'x' and 'y'.

Given equation:- $F\left(\frac{xy}{z}, \frac{x-y}{z}\right)=0$ — (1)

Let $u=\frac{xy}{z}$ $v=\frac{x-y}{z}$ — (2)

$\Rightarrow F(u,v)=0$ — (3) be the surface eqn.

By known theorem, if $F(u,v)=0$ be the surface eqn then the P.D.E. for this surface eqn is

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$\Rightarrow p[u_y v_z - u_z v_y]+q[u_z v_x - u_x v_z] = u_x v_y - u_y v_x$ — (4)

Since $u=\frac{xy}{z}$ $v=\frac{x-y}{z}$

$u_x=\frac{y}{z}$ $v_x=\frac{1}{z}$

$u_y=\frac{x}{z}$ $v_y=-\frac{1}{z}$

$u_z=-\frac{xy}{z^2}$ $v_z=-\frac{(x-y)}{z^2}$

Substitude these values in (4) we get

$$p\left[\frac{x}{z}\left(-\frac{(x-y)}{z^2}\right)+\frac{xy}{z^2}\left(-\frac{1}{z}\right)\right]+q\left[-\frac{xy}{z^2}\left(\frac{1}{z}\right)+\frac{y}{z}\frac{(x-y)}{z^2}\right] = \frac{y}{z}\left(-\frac{1}{z}\right)-\frac{x}{z}\left(\frac{1}{z}\right)$$

$$p\left[\frac{-x^2+\cancel{xy}-\cancel{xy}}{z^3}\right]+q\left[\frac{-\cancel{xy}+\cancel{yx}-y^2}{z^3}\right] = \frac{-y-x}{z^2}$$

$-x^2p-qy^2 = -z(y+x)$

$px^2+qy^2 = zy+zx$

$px^2+qy^2-zy-zx=0$ is the required P.D.E.

7 $f(xy, x+y-z)=0$

Given eqn is $f(xy, x+y-z)=0$ — (1)

Let $u=xy$ $v=x+y-z$ — (2)

$\Rightarrow F(u,v)=0$ — (3) be the surface eqn

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$\Rightarrow p[u_y v_z - u_z v_y]+q[u_z v_x - u_x v_z] = u_x v_y - u_y v_x$ — (4)